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On Liouville's Function

By R. Sherman Lehman

1. Introduction. Liouville's function $\lambda(n)$ is defined by the equation $\lambda(n) = (-1)^r$ where r is the number of prime factors of n, multiple factors being counted according to their multiplicity. Pólya [6] conjectured that the function

$$L(x) = \sum_{n \leq x} \lambda(n)$$

is negative or zero for all $x \geqq 2$, and in fact this is true within the range where this function has been previously calculated. Calculations connected with the present study show that $L(x) \leqq 0$ for $2 \leqq x \leqq 10^6$. Nevertheless Haselgrove [3] has shown that the Pólya conjecture is false and that there are infinitely many integers x for which $L(x) > 0$. However, his method does not furnish explicitly an x for which the conjecture fails; and in fact it does not give an upper bound for the first counterexample. In the present paper we shall describe calculations which lead to the result that $L(906{,}180{,}359) = +1$. We have not found a smaller value of x for which the conjecture fails, but also we have not proved that this is the smallest x greater than 2 for which $L(x)$ is positive.

2. Background and Heuristic Considerations. Liouville's function is connected with the Riemann zeta function by the equation

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}.$$

Let the zeros of $\zeta(s)$ on the line $\operatorname{Re} s = \frac{1}{2}$ be $\rho_n = \frac{1}{2} + i\gamma_n$ $(n = \pm 1, \pm 2, \cdots)$ with $\gamma_{-n} = -\gamma_n$ and let $\gamma_0 = 0$. If it is assumed that these zeros are all simple, then the function $\zeta(2s)/(s\zeta(s))$ has simple poles at $\frac{1}{2} + i\gamma_n$ for $n = 0, \pm 1, \pm 2, \cdots$ with residues

$$\alpha_0 = \frac{1}{\zeta(\frac{1}{2})}, \qquad \alpha_n = \frac{\zeta(2\rho_n)}{\rho_n \zeta'(\rho_n)} \qquad (n = \pm 1, \pm 2, \pm 3, \cdots).$$

Fawaz [1] has obtained an explicit formula for $L(x)$ which is valid if the Riemann hypothesis holds and the zeros of $\zeta(s)$ are simple. Under these assumptions he showed that there is a sequence of numbers T_k, with $k < T_k < k + 1$, for which

$$L(x) = \lim_{k \to \infty} \sum_{|\gamma_n| \leq T_k} \alpha_n x^{\frac{1}{2} + i\gamma_n} + O(1) \tag{1}$$

for $x > 0$.

Let

$$A_T(u) = \sum_{|\gamma_n| \leq T} \alpha_n e^{i\gamma_n u}. \tag{2}$$

Fawaz's result suggests that one study numerically the behavior of $A_T(u)$ for different values of T. Since $A_T(u)$ should be an approximation to $e^{-\frac{1}{2}u}L(e^u)$, one might expect $L(e^u)$ to be positive for a u for which $A_T(u) > 0$.

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In §5 we shall show that if the Riemann hypothesis holds, if the zeros of $\zeta(s)$ are simple, and if a conjectured estimate for $1/\zeta'(\rho_n)$ holds, then

$$\lim_{\omega\to\infty}\left\{\int_{\frac{1}{2}\omega}^{\omega} K_T(u-\omega)e^{-\frac{1}{2}u}L(e^u)\,du-\sum_{|\gamma_n|\leq T}\alpha_n e^{i\gamma_n\omega}\right\}=0, \tag{3}$$

where

$$K_T(t)=\frac{\sin Tt}{\pi t}.$$

This suggests that the sum $A_T(u)$ will represent a smoothing of the function $e^{-\frac{1}{2}u}L(e^u)$ which effectively operates over an interval of width about $2\pi/T$. However, since the kernel $K_T(t)$ is not always positive, the sum does not represent a true smoothing of $e^{-\frac{1}{2}u}L(e^u)$. Thus, if we find a maximum of the sum $A_T(u)$ we cannot always expect that there will be a value of $e^{-\frac{1}{2}u}L(e^u)$ in the vicinity which is as high. Instead it is possible for a rapid falling off of $e^{-\frac{1}{2}u}L(e^u)$ at a distance away of from π/T to $2\pi/T$ to be translated into an extra high peak of $A_T(u)$. On the other hand, high values of $A_T(u)$ for several different choices of T will make such behavior appear less likely.

Haselgrove's disproof of the Pólya conjecture was based on a numerical study of the similar sum

$$A_T^*(u)=\sum_{|\gamma_n|\leq T}\alpha_n\left(1-\frac{|\gamma_n|}{T}\right)e^{i\gamma_n u}.$$

Ingham [5] had shown that for any u_0 .

$$A_T^*(u_0)\leqq\limsup_{u\to\infty}e^{-\frac{1}{2}u}L(e^u).$$

Hence to disprove the Pólya conjecture it was sufficient to find a T and u_0 for which $A_T^*(u_0)>0$. Haselgrove found that $A_{1000}^*(831.847)=0.00495$.

Using an IBM 701 at the University of California, Berkeley, we have independently computed approximations to the zeros of $\zeta(s)$ and the residues α_n and have confirmed Haselgrove's result. We have also obtained a smaller value of u_0 for which $A_{1000}^*(u_0)$ is positive. Our values to 4 decimal places are

$$A_{1000}^*(831.847)=0.0050,\qquad A_{1000}(831.847)=0.0526$$

$$A_{1000}^*(814.492)=0.0782,\qquad A_{1000}(814.492)=0.1102.$$

As a result of a search for smaller values of u for which $A_{1000}(u)>0$ we found

$$A_{1000}^*(79.28)=-0.0418,\qquad A_{1000}(79.28)=0.0075.$$

The number $e^{79.28}$ is still a very large number, and there does not seem to be any more hope of calculating $L(x)$ for $x=e^{79.28}$ than for $x=e^{831.847}$. On the other hand, it is possible to find a method for calculating $L(x)$ at isolated values which is quite feasible with present-day computers for x as large as 10^9. Therefore, we computed $A_T(u)$ for $u=12.5(0.01)20.69$ with $T=100$ and $T=200$, covering approximately the range $2.7\cdot10^5<x<9.7\cdot10^8$. The vicinities of several high points were then selected for further study. The most promising of these was near

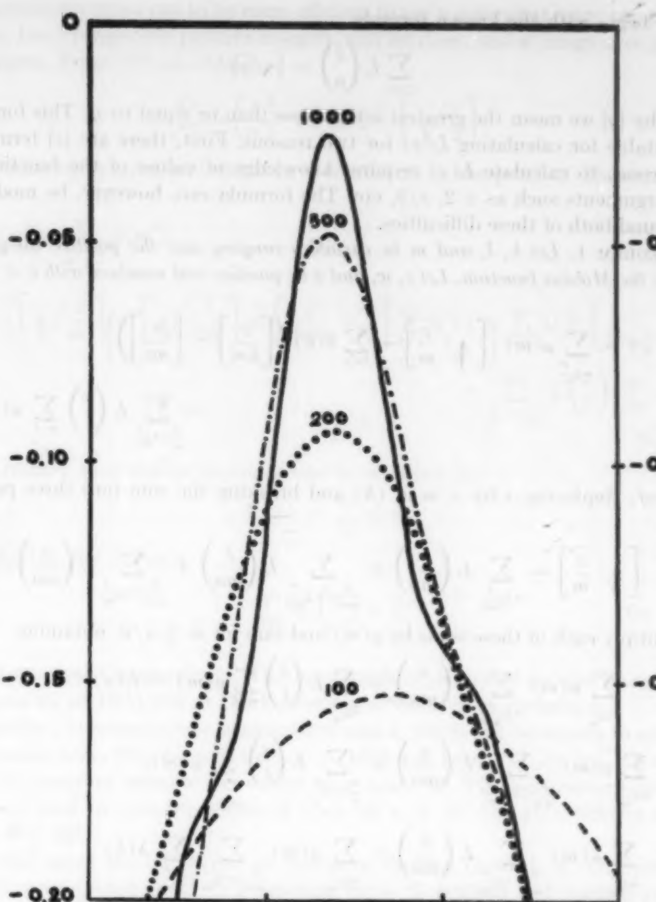

FIG. 1.—The functions $A_T(u)$ for $T = 100, 200, 500, 1000$. The maximum of $A_{1000}(u)$ is near $u = 20.623 = \log(9.05 \cdot 10^8)$.

$u = 20.62$. Figure 1 shows a graph of $A_T(u)$ for u in this vicinity with four different values of T. Although none of the functions $A_T(u)$ graphed is positive there, the strong upward trend suggested the conjecture that $L(x)$ is positive for some x near $9.05 \cdot 10^8$.

3. A Formula for Calculating $L(x)$. A direct calculation of $L(x)$ for all $x \leqq 10^9$ by factoring each number would require far too much machine time to be feasible on the IBM 704—at least 1000 hours. Fortunately it is possible to devise an appropriate method for calculating $L(x)$ at isolated values.

We begin with the known formula

$$\sum_{n \leqq x} L\left(\frac{x}{n}\right) = [\sqrt{x}] \tag{4}$$

where by $[y]$ we mean the greatest integer less than or equal to y. This formula is not suitable for calculating $L(x)$ for two reasons. First, there are $[x]$ terms in it; and second, to calculate $L(x)$ requires knowledge of values of the function L at large arguments such as $x/2$, $x/3$, etc. The formula can, however, be modified to get around both of these difficulties.

THEOREM 1. *Let k, l, and m be variables ranging over the positive integers. Let $\mu(n)$ be the Möbius function. Let v, w, and x be positive real numbers with $v < w < x$. Then*

$$L(x) = \sum_{m \leqq \frac{x}{w}} \mu(m) \left\{ \left[\sqrt{\frac{x}{m}}\right] - \sum_{k < v} \lambda(k) \left(\left[\frac{x}{km}\right] - \left[\frac{x}{mv}\right] \right) \right\} - \sum_{\frac{x}{w} < l \leqq \frac{x}{v}} L\left(\frac{x}{l}\right) \sum_{\substack{m \mid l \\ m \leqq \frac{x}{w}}} \mu(m).$$

Proof: Replacing x by x/m in (4) and breaking the sum into three parts, we obtain

$$\left[\sqrt{\frac{x}{m}}\right] = \sum_{n \leqq \frac{x}{mw}} L\left(\frac{x}{mn}\right) + \sum_{\frac{x}{mw} < n \leqq \frac{x}{mv}} L\left(\frac{x}{mn}\right) + \sum_{\frac{x}{mv} < n \leqq \frac{x}{m}} L\left(\frac{x}{mn}\right).$$

We multiply each of these sums by $\mu(m)$ and sum for $m \leqq x/w$, obtaining

$$\sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{n \leqq \frac{x}{mw}} L\left(\frac{x}{mn}\right) = \sum_{l \leqq \frac{x}{w}} L\left(\frac{x}{l}\right) \sum_{m \mid l} \mu(m) = L(x),$$

$$\sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{\frac{x}{mw} < n \leqq \frac{x}{mv}} L\left(\frac{x}{mn}\right) = \sum_{\frac{x}{w} < l \leqq \frac{x}{v}} L\left(\frac{x}{l}\right) \sum_{\substack{m \mid l \\ m \leqq \frac{x}{w}}} \mu(m),$$

$$\begin{aligned} \sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{\frac{x}{mv} < n \leqq \frac{x}{m}} L\left(\frac{x}{mn}\right) &= \sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{\frac{x}{mv} < n \leqq \frac{x}{m}} \sum_{k \leqq \frac{x}{mn}} \lambda(k) \\ &= \sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{k < v} \lambda(k) \sum_{\frac{x}{mv} < n \leqq \frac{x}{km}} 1 \\ &= \sum_{m \leqq \frac{x}{w}} \mu(m) \sum_{k < v} \lambda(k) \left(\left[\frac{x}{km}\right] - \left[\frac{x}{mv}\right] \right). \end{aligned}$$

Rearrangement of terms then yields the theorem.

We observe that if $v \approx x^{1/3}$ and $w \approx x^{2/3}$ then the number of operations is proportional to $x^{2/3}$ if one has available a table of $\lambda(n)$ for $n \leqq w$, a table of

$$\xi(l) = \sum_{\substack{m \mid l \\ m \leqq \frac{x}{w}}} \mu(m) \tag{5}$$

for $l \leqq x/v$, and a table of $\mu(m)$ for $m \leqq x/w$.

In practice it turns out to be more efficient to use a somewhat more complicated formula. Let k range over positive integers, and let l', m', and n' range over positive *odd* integers. From (4) we obtain

$$\sum_{n' \leqq x} L\left(\frac{x}{n'}\right) = [\sqrt{x}] - \left[\sqrt{\frac{x}{2}}\right].$$

If x is replaced by x/m' and the sum is treated as in the proof of Theorem 1, we obtain

$$\begin{aligned} L(x) = \sum_{m' \leqq \frac{x}{w}} \mu(m') & \\ \cdot \Bigg\{\left[\sqrt{\frac{x}{m'}}\right] - \left[\sqrt{\frac{x}{2m'}}\right] + \left[\frac{x}{2m'v} - \frac{1}{2}\right] \sum_{k<v} \lambda(k) - \sum_{k<v} \lambda(k) \left[\frac{x}{2km'} - \frac{1}{2}\right]\Bigg\} & \quad (6) \\ - \sum_{\frac{x}{w} < l' \leqq \frac{x}{v}} L\left(\frac{x}{l'}\right) \sum_{\substack{m' \mid l' \\ m' \leqq \frac{x}{w}}} \mu(m'). & \end{aligned}$$

We remark that similar formulas can be obtained for

$$M(x) = \sum_{n \leqq x} \mu(n)$$

by modifying the formula

$$\sum_{n \leqq x} M\left(\frac{x}{n}\right) = 1 \qquad \text{for } x \geqq 1.$$

4. Numerical Computations. The computations described in this section were performed on an IBM 704 at the University of California, Berkeley.

Formula (6) contains two parameters v and w which can be chosen to minimize computation time. We chose to fix $x/w = 1000$ and take $v = (10^{-3}x)^{1/2}$ in order to make the program suitable for values of x near 10^9. Two preliminary programs were then used to compute tables of $\lambda(n)$ for $n \leqq 10^6$ and $\xi(l)$, defined by (5), for odd $l < 10^6$.

In 1955 using the ORDVAC at Aberdeen Proving Ground, W. G. Spohn and the author computed Liouville's function for $n \leqq 802{,}000$ and verified that the Pólya conjecture holds up to this limit. In the present computation, the same method was used to obtain a table of $\lambda(n)$ for $n \leqq 10^6$. If one is given a table of $\lambda(n)$ for $n \leqq N/2$, then the following sieving process will allow the extension of the table to N. One begins by entering -1 as the value for each integer from $N/2$ to N. One then considers in turn each of the primes $p \leqq \sqrt{N}$, and one runs through the multiples of p among the integers from $N/2$ to N. If n is such a multiple, one sets $\lambda(n) = -\lambda(n/p)$ after erasing the value already recorded for n. When this is done for all multiples of primes $\leqq \sqrt{N}$, the table is complete. For a machine with enough storage space to hold the table of $\lambda(n)$ for $n \leqq N/2$, this method is much more efficient than factoring each integer in succession.

The table of $\lambda(n)$ was placed on magnetic tape with each value taking up one bit. This table was then summed to compute $L(x)$ for $x = 100(100)1{,}000{,}000$. The values for $x = 1000(1000)802{,}000$ were compared with the ORDVAC com-

putation and all were found to agree. Also a comparison for $x = 100{,}000(100{,}000)$-$600{,}000$ was made with a computation of D. H. Lehmer. Finally as a further check, the formula (6) was later used to compute $L(x)$ for $x = 200{,}000(200{,}000)1{,}000{,}000$. The values found agreed with those obtained by summing the table of $\lambda(n)$. (The circularity here is only apparent; in these cases the formula (6) makes no use of the table for $\lambda(n)$ beyond $n = 1000$). To compute the table for $\lambda(n)$ and sum it to obtain $L(x)$ required approximately 30 minutes of machine time.

A table of the function $\xi(l)$ was also needed for odd $l < 10^6$. By a combinatorial argument, which is easy but requires consideration of a number of cases, one can show that if $l < 10^6$, then $-7 \leqq \xi(l) \leqq 7$. Hence each value requires just 4 bits of storage. Again the table was put on magnetic tape. The method for computation was quite straightforward. Each odd square-free number $m < 1000$ was considered, and for each of its multiples $\mu(m)$ was added to a corresponding accumulator. A check of the accuracy of the computation was made by using the formula

$$\sum_{l' \leq x}{}^{!}\, \xi(l') = \sum_{m' < 10^3} \mu(m') \left[\frac{x}{2m'} + \frac{1}{2} \right]$$

with $x = 10^6$ and l' and m' running over positive odd integers. The program for $\xi(l)$ required approximately 20 minutes.

Next a program for computing $L(x)$ by the formula (6) was constructed. The odd square-free numbers $m' < 1000$ together with the values $\mu(m')$ were stored as constants. Newton's method was used to compute $[\sqrt{y}]$ with especial care taken to avoid error due to round-off when $\sqrt{y}$ is an integer. The tables of $\lambda(n)$ and $\xi(l)$ on magnetic tapes were used as inputs, and $L(x/l')$ was obtained by summing the table of $\lambda(n)$. To obtain $L(x)$ for a value of x near $9 \cdot 10^8$ required approximately 16 minutes.

Table 1 contains values of $L(x)$ computed in connection with the search for a positive value. The order of computation is indicated in the last column. The values of x were chosen partly by guess and partly by heuristic considerations based on (3). There seem to be two separate peaks which for the functions $A_T(u)$ were smoothed into a single maximum. After we found a positive value on the seventh trial, it did not seem worthwhile to pursue an investigation of the other peak. Hence we do not know whether the maximum for the other peak is also positive.

TABLE 1

x	$L(x)$	Order of computation
903 000 000	−952	3
904 000 000	−1144	2
905 000 000	−1902	1
906 000 000	−584	4
906 170 000	−230	10
906 200 000	90	8
906 300 000	648	9
906 400 000	708	7
906 470 000	226	11
906 500 000	−120	5
907 000 000	−1920	6

Some other values of $L(x)$ computed by means of the main program are listed in Table 2. The values for $x \leq 10^6$ are all confirmed by agreement with those obtained by directly summing the table of $\lambda(n)$.

The values of $L(x)$ for $x = 10^6$ and $4 \cdot 10^6$ are confirmed by results of a hand computation of D. H. Lehmer using the formula

$$L(x) = \sum_{k \leq g} M\left(\frac{x}{k^2}\right) + \sum_{l \leq \frac{x}{g^2}} \mu(l)\left[\sqrt{\frac{x}{l}}\right] - M\left(\frac{x}{g^2}\right)\left[\sqrt{\frac{x}{g^2}}\right]$$

with g chosen to minimize computation. In this computation the values for $M(x/k^2)$ were taken from a corrected version of von Sterneck's tables of $M(x)$ (see [7]) with the following values differing from those given by von Sterneck: $M(444{,}444) = -37$, $M(10^6) = 212$, $M(4 \cdot 10^6) = 192$. The first two of these values were obtained by factoring all numbers $\leq 10^6$. The value for $4 \cdot 10^6$ was obtained by making small adjustments of von Sterneck's computation which are required because of errors in the earlier tables.

After finding positive values of $L(x)$ we next took up the question of determining zeros of $L(x)$. The results given in Table 1 indicated that such zeros must occur between 906,170,000 and 906,200,000 and between 906,470,000 and 906,500,000. Consequently a program was constructed to factor all numbers in these ranges. This program required approximately one minute for each 1000 numbers factored. As a byproduct of this computation we obtained a further check of the program for computing $L(x)$ at isolated values.

One of the results of this computation was a listing of all zeros of $L(x)$ in the intervals from 906,170,000 to 906,200,000 and from 906,470,000 to 906,500,000. In all, 167 zeros of the function $L(x)$ were found in these intervals. We list the first 16 occurring in the first interval:

906180358, 906180362, 906180364, 906180366, 906180370,
906180374, 906180376, 906180386, 906180388, 906180390,
906180518, 906180520, 906180524, 906180534, 906180536,
906180554.

There are 34 zeros from 906,192,698 to 906,193,478 inclusive; 22 zeros from 906,-194,914 to 906,195,298; 19 zeros from 906,195,986 to 906,196,098; 15 zeros from 906,477,702 to 906,477,936; 43 zeros from 906,486,640 to 906,487,288; and 18 zeros from 906,487,932 to 906,488,080.

TABLE 2

x	$L(x)$	x	$L(x)$
200 000	−294	10 400 000	−394
400 000	−460	10 410 000	−330
600 000	−802	10 420 000	−384
800 000	−600	10 430 000	−300
1 000 000	−530	10 440 000	−292
4 000 000	−1098	10 450 000	−522
		10 460 000	−588
		453 200 000	−27088

The first value of x greater than 906,170,000 for which $L(x)$ is positive was found to be 906,180,359. We, of course, are not able to say whether this is the smallest x greater than 2 for which $L(x)$ is positive. To decide this question without the use of essentially new ideas might very well require an enormous amount of computation.

5. Derivation of an Explicit Formula. In this section we give a derivation of equation (3), which was used heuristically in finding where $L(x)$ is positive; this derivation proceeds from several unproved assumptions. We shall assume that the Riemann hypothesis holds and that the zeros $\rho_n = \frac{1}{2} + i\gamma_n$ $(n = \pm 1, \pm 2, \cdots)$ of $\zeta(s)$ are all simple. In addition we shall assume that there is a real number $\nu < 1$ such that

$$\frac{1}{\zeta'(\rho_n)} = O(\rho_n^{\nu}) \qquad (n = \pm 1, \pm 2, \cdots). \tag{7}$$

Numerical evidence makes this conjectured estimate appear quite plausible. The twelve largest values of $|1/\zeta'(\rho_n)|$ for $0 < \gamma_n < 1000$ are listed in Table 3.

As in §2 let $K_T(t) = (\sin Tt)/(\pi t)$.

Lemma 1. *If $R > 0$ and $T > 0$ and γ is a real number, then*

$$\int_{-R}^{R} K_T(t)e^{i\gamma t}\,dt = \begin{cases} 1 + O\left(\dfrac{1}{R(T - |\gamma|)}\right) & \text{if } |\gamma| < T, \\[2ex] O\left(\dfrac{1}{R(|\gamma| - T)}\right) & \text{if } |\gamma| > T. \end{cases}$$

Proof: We have

$$\begin{aligned} \int_{-R}^{R} K_T(t)e^{i\gamma t}\,dt &= \frac{2}{\pi}\int_0^R \frac{\sin Tt}{t}\cos \gamma t\,dt \\ &= \frac{1}{\pi}\int_0^R \frac{\sin (T+\gamma)t}{t}\,dt + \frac{1}{\pi}\int_0^R \frac{\sin (T-\gamma)t}{t}\,dt \\ &= \frac{1}{\pi}\operatorname{Si}(R(T+\gamma)) + \frac{1}{\pi}\operatorname{Si}(R(T-\gamma)), \end{aligned}$$

where

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt = \begin{cases} \dfrac{\pi}{2} + O\left(\dfrac{1}{x}\right) & \text{for } x > 0, \\[2ex] -\dfrac{\pi}{2} + O\left(\dfrac{1}{x}\right) & \text{for } x < 0. \end{cases}$$

The conclusion of the lemma follows immediately.

We shall also use the following estimate for $R > 0$ and $T > 0$:

$$\begin{aligned} \int_{-R}^{R} |K_T(t)|\,dt &= \frac{2}{\pi}\int_0^{RT} \frac{|\sin t|}{t}\,dt < \int_0^1 dt + \int_1^{RT}\frac{dt}{t} \\ &= 1 + \log (RT). \end{aligned} \tag{8}$$

TABLE 3

n	γ_n	$\lvert 1/\zeta'(\frac{1}{2} + i\gamma_n) \rvert$
1	14.134725	1.2608
2	21.022040	0.8796
72	185.598784	0.8109
135	294.965370	0.8029
298	540.213166	0.8357
299	540.631390	0.8892
363	630.473887	0.8334
364	630.805781	0.9106
436	728.405482	0.8371
437	728.758750	0.8491
606	946.765842	0.9744
607	947.079183	0.9914

We are now ready to derive (3). We shall assume throughout that $R > 0$, $T > 0$, and $T \neq \gamma_n$ for $n = 1, 2, 3, \cdots$. The remainder of the notation is explained in §2.

Fawaz [2, p. 284] has shown (assuming the Riemann hypothesis and the simplicity of the zeros) that if u is restricted to a finite interval, then there is a positive constant C independent of k and u such that

$$\Big| \sum_{|\gamma_n| \leq T_k} \alpha_n e^{i\gamma_n u} \Big| < C.$$

Also $| K_T(u - \omega) | \leq T/\pi$. Consequently we can apply the Lebesgue bounded convergence theorem to obtain

$$\lim_{k\to\infty} \int_{\omega-R}^{\omega+R} \Big(K_T(u - \omega) \sum_{|\gamma_n|\leq T_k} \alpha_n e^{i\gamma_n u}\Big)\, du = \int_{\omega-R}^{\omega+R} K_T(u - \omega) \Big(\lim_{k\to\infty} \sum_{|\gamma_n|\leq T_k} \alpha_n e^{i\gamma_n u}\Big)\, du.$$

Equation (1) can be written in the form

$$e^{-\frac{1}{2}u} L(e^u) = \lim_{k\to\infty} \sum_{|\gamma_n|\leq T_k} \alpha_n e^{i\gamma_n u} + O(e^{-\frac{1}{2}u}).$$

Hence we obtain

$$\tag{9} \begin{aligned} \int_{\omega-R}^{\omega+R} K_T(u-\omega) e^{-\frac{1}{2}u} L(e^u)\, du = \lim_{k\to\infty} \sum_{|\gamma_n|\leq T_k} \Big(\alpha_n e^{i\gamma_n \omega} \int_{-R}^{R} K_T(t) e^{i\gamma_n t}\, dt\Big) \\ + \int_{\omega-R}^{\omega+R} K_T(u-\omega)\, O(e^{-\frac{1}{2}u})\, du. \end{aligned}$$

If $T_k \geq 2T$, then by Lemma 1 we have

$$\tag{10} \begin{aligned} \sum_{|\gamma_n|\leq T_k} \alpha_n e^{i\gamma_n \omega} \int_{-R}^{R} K_T(t) e^{i\gamma_n t}\, dt = \sum_{|\gamma_n|\leq T} \alpha_n e^{i\gamma_n \omega} \\ + O\Big(\sum_{0\leq\gamma_n\leq 2T} \frac{|\alpha_n|}{R\,|T - |\gamma_n||}\Big) + O\Big(\sum_{2T<\gamma_n\leq T_k} \frac{|\alpha_n|}{R(|\gamma_n|/2)}\Big). \end{aligned}$$

Since $\zeta(1 + it) = O(\log t)$ for $t > 2$, we obtain the following estimate from the assumption (7):

$$\frac{|\alpha_n|}{|\gamma_n|} = \frac{|\zeta(2\rho_n)|}{|\gamma_n \rho_n \zeta'(\rho_n)|} = O(\gamma_n^{\nu-2} \log \gamma_n)$$

for $n = 1, 2, 3, \cdots$. Since the series $\sum_{\gamma_n > 0} \gamma_n^{-\beta}$ converges for $\beta > 1$, (see [4, p. 57]), it follows that the series

$$\sum_{\gamma_n > 0} \frac{|\alpha_n|}{\gamma_n}$$

converges; hence the last term on the right-hand side of (10) is $O(1/R)$ uniformly in k. For *fixed* T the second term on the right-hand side of (10) is also $O(1/R)$ since it is a finite series.

Estimating the last term on the right-hand side of (9), we obtain

$$\int_{\omega-R}^{\omega+R} K_T(u - \omega)\, O(e^{-\frac{1}{2}u})\, du = O(e^{-\frac{1}{2}(\omega-R)}) \int_{-R}^{R} |K_T(t)|\, dt$$
$$= O(e^{-\frac{1}{2}(\omega-R)} \log (RT))$$

for $R > 2/T$.

Combining (9) and (10) and letting $k \to \infty$, we conclude that for fixed T,

$$\int_{\omega-R}^{\omega+R} K_T(u - \omega) e^{-\frac{1}{2}u} L(e^u)\, du = \sum_{|\gamma_n| \leqq T} \alpha_n e^{i\gamma_n \omega} + O\left(\frac{1}{R}\right) + O(e^{-\frac{1}{2}(\omega-R)} \log (RT)),$$

provided $R > 2/T$. Equation (3) can now be obtained by taking $R = \omega/2$ and letting $\omega \to \infty$.

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On the Conjecture of Hardy & Littlewood concerning the Number of Primes of the Form $n^2 + a$

By Daniel Shanks

1. Introduction. In a famous paper, [1], Hardy and Littlewood developed a number of conjectures concerning the twin primes, the Goldbach problem, and other unsettled questions. One of these, Conjecture F, concerned the number of primes of the form $Am^2 + Bm + C$. We reword this conjecture, and at the same time reduce its generality somewhat, as follows:

CONJECTURE. *If a is an integer which is not a negative square, $a \neq -k^2$, and if $P_a(N)$ is the number of primes of the form $n^2 + a$ for $1 \leq n \leq N$, then*

$$P_a(N) \sim \frac{1}{2} h_a \int_2^N \frac{dn}{\log n} \tag{1}$$

where the constant h_a is the infinite product

$$h_a = \prod_{w \nmid a}^{\infty} \left(1 - \left(\frac{-a}{w}\right) \frac{1}{w-1}\right) \tag{2}$$

taken over all odd primes, w, which do not divide a, and for which $(-a/w)$ is the Legendre symbol.

In the trivial cases, $a = -k^2$, since $(k^2/w) = +1$ for every w, we have $h_a = 0$ on the one hand, and on the other there can be at most one prime of the form $n^2 - k^2 = (n - k)(n + k)$. For any other a, $h_a > 0$, and the conjecture indicates that there are infinitely many primes. But for no a has this been proven.

In particular, for $a = 1$, since $(-1/w)$ equals $+1$ or -1 according as $w = 4m + 1$ or $4m - 1$, we have

$$h_1 = (1 + \tfrac{1}{2})(1 - \tfrac{1}{4})(1 + \tfrac{1}{6})(1 + \tfrac{1}{10})(1 - \tfrac{1}{12}) \cdots = 1.37281346 \cdots \tag{3}$$

and therefore (1) implies that

$$P_1(N) \sim 0.68640673 \int_2^N \frac{dn}{\log n}. \tag{4}$$

A. E. Western [2] verified that the number of primes of the form $n^2 + 1$ agreed well with the right side of (4) up to $N = 15{,}000$.

In a recent paper [3] a sieve method was developed for factoring numbers of the form $n^2 + 1$, and more generally of the form $n^2 + a$, and it was shown that the good agreement in (4) continues to hold out to $N = 180{,}000$; $(N^2 + 1 = 32{,}400{,}000{,}001)$. This verification, however, was not applied to (4) directly but to the related formula, (7), given below.

Let $\bar{\pi}_a(N)$ be the number of odd primes, q, which are $\leq N$, which do not divide

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a, and for which $(-a/q) = -1$. These are the primes which never divide $n^2 + a$. It is well known that

$$\bar{\pi}_a(N) \sim \frac{1}{2}\int_2^N \frac{dn}{\log n} \tag{5}$$

and therefore (1) can be rewritten as

$$\frac{P_a(N)}{\bar{\pi}_a(N)} \sim h_a. \tag{6}$$

Likewise (4) can be rewritten as

$$\frac{P_1(N)}{\bar{\pi}_1(N)} \sim 1.37281346 \cdots. \tag{7}$$

Since, in [3], we had $P_1(180{,}000) = 11223$, $\bar{\pi}_1(180{,}000) = 8178$, and $11223/8178 = 1.37234$, the agreement with the right side of (7) was even better than could be expected.

It is clear that the $\bar{\pi}_a(N)$ in (6) could be replaced by the asymptotically equal $\frac{1}{2}\pi(N)$ or by $\overset{+}{\pi}_a(N)$, (for the latter number we count the p's such that $(-a/p) = +1$). But (6) as it stands is to be preferred for two reasons. First, $\bar{\pi}_a(N)$ is generally much closer to $\frac{1}{2}\int_2^N dn/\log n$ than are either of the other two counts. See [4, sec. 10 and Table 7] for a discussion of the case $a = 1$. Second, the ratio in (6) has a simple geometric interpretation in the algebraic number field $R(\sqrt{-a})$. See [3, p. 82] for a discussion of the case $a = 1$, the Gauss plane.

In the present paper [5] we first develop an interesting and rapidly converging formula for computing the h_a and we tabulate these constants for $a = -4(1)4$. We then present short tables of $P_a(N)$ and $\bar{\pi}_a(N)$ for $a = \pm 2, \pm 3, +4$, and for $N = 10{,}000(10{,}000)180{,}000$ which show that (6) also gives good agreement in these five cases. Finally we present an elementary (sieve) argument which makes it plausible that the Hardy-Littlewood conjecture is true for every a. Further, an analysis of this computation enables us to isolate the essential difficulty in obtaining a proof.

2. The Right Side of (6). To compute the h_a we will want the following

LEMMA. *For* $|x| < \frac{1}{2}$,

$$\frac{1}{1-2x} = \prod_{s=1}^{\infty}\left(\frac{1+x^s}{1-x^s}\right)^{b(s)} \tag{8}$$

where the exponents $b(s)$ *are given by* $b(1) = b(2) = b(3) = 1, b(4) = 2, b(5) = 3, b(6) = 5$, *and, in general, if* d *is an odd divisor of* s *and* $\mu(d)$ *is its Möbius function, then*

$$b(s) = \frac{1}{2s}\sum_d \mu(d)2^{s/d}. \tag{9}$$

Examples of (9): A.) If $s = p$, an odd prime, $d = 1$ or $d = p$ and [6]

$$b(p) = (2^p - 2)/2p = (2^{p-1} - 1)/p. \tag{9a}$$

B.) If $s = 2^k$, then d can only equal 1 and

$$b(s) = 2^{s-1}/s. \tag{9b}$$

Therefore $b(7) = 9$ and $b(8) = 16$.

PROOF OF THE LEMMA. After taking the logarithm of both sides of (8),

$$-\ln(1 - 2x) = \sum_{s=1}^{\infty} b(s) \ln[(1 + x^s)/(1 - x^s)], \tag{10}$$

we expand both sides in Maclaurin series and identify the corresponding coefficients. This yields the condition, for $s = 2^k m$, with m odd,

$$2^{s-1} = \sum_{d/m} \frac{s}{d} b\left(\frac{s}{d}\right). \tag{11}$$

Now applying the Möbius inversion formula we obtain (9). Since from (11) we also have $b(s) \leq 2^s/2s$ it follows that (10) converges if $|x| < \frac{1}{2}$ and the steps may be reversed to yield (8).

Now for any $a \neq -k^2$ let p_i be the odd primes such that $(-a/p) = +1$, let q_i be the odd primes such that $(-a/q) = -1$, and let $r_1 = 2, r_2, r_3, \cdots, r_c$ be the (finite number of) primes which divide $2a$. Further, for $s = 1, 2, 3, \cdots$, let

$$L_a(s) = \left[\prod_{p,q}\left(1 - \frac{1}{p^s}\right)\left(1 + \frac{1}{q^s}\right)\right]^{-1}, \tag{12}$$

the product being taken over the p's and q's in numerical order. Finally for $s = 2, 3, 4, \cdots$, let

$$\zeta_a(s) = \zeta(s) \prod_{i=1}^{c} (1 - r_i^{-s}) \tag{13}$$

where $\zeta(s)$ is the Riemann zeta function.

THEOREM. *If*

$$f_a^{(0)} = \zeta_a(2)/L_a(1) \text{ and } K_a^{(0)}(s) = \zeta_a(2s)/L_a(s)\zeta_a(s) \tag{14}$$

for $s = 2, 3, 4, \cdots$, *then*

$$h_a = f_a^{(0)} \cdot \prod_{s=2}^{\infty} [K_a^{(0)}(s)]^{b(s)}, \tag{15}$$

where $b(s)$ *is given by* (9). *More generally, for more rapid convergence, we may select a positive integer* u *and define*

$$f_a^{(u)} = f_a^{(0)} \prod_{i=1}^{u}\left(1 - \frac{2}{p_i(p_i - 1)}\right) = f_a^{(0)} \prod_{i=1}^{u}\left(1 - \frac{2}{p_i}\right)\left(\frac{p_i + 1}{p_i - 1}\right), \tag{16}$$

and

$$K_a^{(u)}(s) = K_a^{(0)}(s) \prod_{i=1}^{u}\left(1 + \frac{2}{p_i^s - 1}\right) = K_a^{(0)}(s) \prod_{i=1}^{u}\left(\frac{p_i^s + 1}{p_i^s - 1}\right). \tag{17}$$

Then for every $u = 0, 1, 2, \cdots$,

$$h_a = f_a^{(u)} \prod_{s=2}^{\infty} [K_a^{(u)}(s)]^{b(s)}. \tag{18}$$

Proof. For every $s = 2, 3, 4, \cdots$,

$$\zeta(s) = \left[\prod_{p,q,r}\left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{q^s}\right)\left(1 - \frac{1}{r^s}\right)\right]^{-1}$$

and we easily verify that

$$1 = K_a^{(0)}(s) \prod_p \left(\frac{p^s + 1}{p^s - 1}\right). \tag{19}$$

We likewise find that

$$h_a = f_a^{(0)} \prod_p \left(1 - \frac{2}{p}\right)\left(\frac{p + 1}{p - 1}\right) \tag{20}$$

so for any positive integer m, we have from (19) and (20)

$$h_a = f_a^{(u)} \prod_{s=2}^{m} [K_a^{(u)}(s)]^{b(s)} \cdot \prod_{i=u+1}^{\infty} \left(1 - \frac{2}{p_i}\right)\left(\frac{p_i + 1}{p_i - 1}\right) \cdot \prod_{s=2}^{m} \prod_{i=u+1}^{\infty} \left(\frac{p_i^s + 1}{p_i^s - 1}\right)^{b(s)}.$$

Since m is finite the order of the products may be changed to give

$$h_a = f_a^{(u)} \prod_{s=2}^{m} [K_a^{(u)}(s)]^{b(s)} \cdot \prod_{i=u+1}^{\infty} \left(1 - \frac{2}{p_i}\right) \cdot \prod_{s=1}^{m} \left(\frac{p_i^s + 1}{p_i^s - 1}\right)^{b(s)}.$$

Now every $p > 2$, and we may therefore use (8) with $x = 1/p_i$ to obtain

$$h_a = f_a^{(u)} \prod_{s=2}^{m} [K_a^{(u)}(s)]^{b(s)} \cdot \prod_{i=u+1}^{\infty} \prod_{s=m+1}^{\infty} \left(\frac{p_i^s - 1}{p_i^s + 1}\right)^{b(s)}.$$

But it may be readily seen that the double infinite product on the right converges (monotonically increasing) to 1 as $m \to \infty$, and it thus follows that the right side of (18) converges (monotonically decreasing) to h_a as $m \to \infty$.

The computation of the h_a from (18) requires knowledge of the $L_a(s)$. Now every $L_a(s)$ has a Dirichlet series

$$L_a(s) = \sum_{n=1}^{\infty} d_n(a) n^{-s}$$

with real periodic coefficients. Specifically we have

$$\begin{aligned}
L_1(s) &= 1 - 3^{-s} + 5^{-s} - 7^{-s} + - + -, \\
L_2(s) &= 1 + 3^{-s} - 5^{-s} - 7^{-s} + + - -, \\
L_{-2}(s) &= 1 - 3^{-s} - 5^{-s} + 7^{-s} + - - +, \\
L_3(s) &= 1 - 5^{-s} + 7^{-s} - 11^{-s} + - + -, \\
L_{-3}(s) &= 1 - 5^{-s} - 7^{-s} + 11^{-s} + - - +, \\
L_4(s) &= 1 - 3^{-s} + 5^{-s} - 7^{-s} + - + -.
\end{aligned} \tag{21}$$

The $L_a(1)$, which enter into $f_a^{(0)}$ as defined by eq. (14), may be obtained in closed form by use of Gauss sums and Fourier series, [7]. Specifically, for $a > 0$ we have the simple

$$L_a(1) = \frac{\pi}{2\sqrt{a}} q_a \tag{22}$$

where the q_a for $1 \leqq a \leqq 100$ are listed in Table 1.

TABLE 1

a	g_a	a	g_a	a	g_a	a	g_a
1	$\frac{1}{2}$	26	6	51	6	76	6
2	1	27	3	52	4	77	8
3	1	28	2	53	6	78	4
4	1	29	6	54	6	79	5
5	2	30	4	55	4	80	8
6	2	31	3	56	8	81	6
7	1	32	4	57	4	82	4
8	2	33	4	58	2	83	9
9	2	34	4	59	9	84	8
10	2	35	6	60	4	85	4
11	3	36	4	61	6	86	10
12	2	37	2	62	8	87	6
13	2	38	6	63	4	88	4
14	4	39	4	64	4	89	12
15	2	40	4	65	8	90	8
16	2	41	8	66	8	91	6
17	4	42	4	67	3	92	6
18	2	43	3	68	8	93	4
19	3	44	6	69	8	94	8
20	4	45	4	70	4	95	8
21	4	46	4	71	7	96	8
22	2	47	5	72	4	97	4
23	3	48	4	73	4	98	8
24	4	49	4	74	10	99	6
25	2	50	6	75	6	100	4

TABLE 2

a	h_a
−4	0
−3	1.38342429
−2	1.85005441
−1	0
0	0
1	1.37281346
2	0.71306310
3	1.12073275
4	1.37281346

The $L_a(1)$ for negative a are a little more complicated and will not be listed here. As regards $L_a(s)$ for other values of s, $L_1(s)$ is a well known function, but except for a few scattered results, [8], values of the other L's do not seem to have been published. J. W. Wrench, Jr. has computed unpublished tables of $L_a(s)$ for $a = \pm 2$ and ± 3. With his permission the author used these tables, together with (18), to compute the four corresponding values of h_a in Table 2. The remaining entries, $h_{-4} = h_{-1} = h_0 = 0$ and $h_4 = h_1$, are trivial.

The variation of the h_a in Table 2 is notable. For example, there should be

more than two and one-half times as many primes of the form $n^2 - 2$ as of the form $n^2 + 2$. As a side remark, we note from (15) that $f_a^{(0)} = 2\zeta_a(2)\sqrt{a}/\pi q_a$ is the leading factor of h_a. Thus for $a > 0$, $n^2 + a$ will therefore have few or many primes according as q_a is large or small (relative to $2\sqrt{a}/\pi$). From Table 1 we see that there will be few primes for $a = 2, 5, 11, 14, 26, 41, 89$, and 194, ($q_{194} = 20$) and there will be many primes for $a = 7, 37, 58$, and 163, ($q_{163} = 3$). The famous function of Euler, $n^2 + n + 41$, equals $\frac{1}{4}[(2n + 1)^2 + 163]$ and its well-known richness in primes is thus closely related to the small value of q_{163}. This, in turn, is related in class number theory to the unique factorization of the integers in the algebraic number field $R(\sqrt{-163})$.

3. The Left Side of (6). Tables of $P_a(N)$ and $\bar{\pi}_a(N)$ for $a = \pm 2, \pm 3, +4$, and $N = 100k$ ($k = 1, 2, \cdots, 1800$) were computed with an IBM 704 program based on the sieve method and the p-adic square roots of $-a$, [3, sec. 9]. At the same time the prime divisors of $n^2 + a$ which do not exceed N were counted, and from these counts the values of $\bar{\pi}_a(N)$ are easily obtained. Summaries of these results are given in Tables 3, 4, and 5. In the last of these, the results for $a = 4$ are compared with the previous results [3] for $a = 1$.

4. Both Sides of (6). In Figure 1 we plot $P_a(N)/\bar{\pi}_a(N)$ versus N together with the conjectured limits, h_a, for $a = \pm 2$ and ± 3. The cases $a = 1$ and $a = 4$, (which should be asymptotically equal since $h_1 = h_4$), are not included in this figure for clarity. If included, these two graphs would intertwine that for the case $a = -3$.

5. An Elementary Interpretation. The over-all impression of the foregoing results is that (6) and its equivalent (1) are almost surely true for $a = 1, \pm 2, \pm 3, 4$.

TABLE 3

N	$P_2(N)$	$\bar{\pi}_2(N)$	$P_2(N)/\bar{\pi}_2(N)$	$P_{-2}(N)$	$\bar{\pi}_{-2}(N)$	$P_{-2}(N)/\bar{\pi}_{-2}(N)$
10000	446	622	0.6737	1153	625	1.8448
20000	817	1134	0.7205	2140	1140	1.8772
30000	1180	1632	0.7230	3087	1631	1.8927
40000	1494	2117	0.7057	3977	2112	1.8830
50000	1821	2580	0.7058	4824	2587	1.8647
60000	2160	3051	0.7080	5643	3041	1.8556
70000	2489	3478	0.7156	6464	3481	1.8569
80000	2823	3942	0.7161	7296	3927	1.8579
90000	3139	4378	0.7170	8083	4374	1.8480
100000	3422	4798	0.7132	8888	4808	1.8486
110000	3721	5229	0.7116	9681	5242	1.8468
120000	4027	5649	0.7129	10500	5682	1.8479
130000	4347	6090	0.7138	11304	6117	1.8480
140000	4652	6516	0.7139	12086	6533	1.8500
150000	4966	6945	0.7150	12828	6956	1.8442
160000	5250	7347	0.7146	13628	7362	1.8511
170000	5522	7767	0.7110	14397	7763	1.8546
180000	5847	8192	0.7138	15134	8184	1.8492

TABLE 4

N	$P_3(N)$	$\bar{\pi}_3(N)$	$P_3(N)/\bar{\pi}_3(N)$	$P_{-3}(N)$	$\bar{\pi}_{-3}(N)$	$P_{-3}(N)/\bar{\pi}_{-3}(N)$
10000	711	616	1.1542	850	620	1.3710
20000	1302	1136	1.1461	1569	1139	1.3775
30000	1851	1633	1.1335	2238	1637	1.3671
40000	2378	2112	1.1259	2903	2108	1.3771
50000	2920	2575	1.1340	3550	2577	1.3776
60000	3428	3041	1.1273	4168	3030	1.3756
70000	3967	3490	1.1367	4796	3466	1.3837
80000	4463	3937	1.1336	5442	3935	1.3830
90000	4941	4373	1.1299	6049	4374	1.3829
100000	5426	4806	1.1290	6664	4819	1.3829
110000	5917	5233	1.1307	7253	5247	1.3823
120000	6410	5665	1.1315	7874	5673	1.3880
130000	6873	6105	1.1258	8491	6097	1.3927
140000	7337	6532	1.1232	9073	6524	1.3907
150000	7823	6940	1.1272	9663	6950	1.3904
160000	8302	7361	1.1278	10236	7363	1.3902
170000	8781	7768	1.1304	10799	7765	1.3907
180000	9240	8195	1.1275	11354	8200	1.3846

TABLE 5

N	$P_4(N)$	$\bar{\pi}_4(N) = \bar{\pi}_1(N)$	$P_4(N)/\bar{\pi}_4(N)$	$P_1(N)$	$P_1(N)/\bar{\pi}_1(N)$	$P_1(N)/P_4(N)$
10000	870	619	1.4055	841	1.3586	0.967
20000	1554	1136	1.3680	1559	1.3724	1.003
30000	2216	1633	1.3570	2268	1.3889	1.023
40000	2838	2117	1.3406	2952	1.3944	1.040
50000	3459	2583	1.3391	3613	1.3988	1.045
60000	4083	3038	1.3440	4252	1.3996	1.041
70000	4690	3485	1.3458	4888	1.4026	1.042
80000	5281	3933	1.3427	5513	1.4017	1.044
90000	5903	4364	1.3527	6084	1.3941	1.031
100000	6517	4808	1.3554	6656	1.3844	1.021
110000	7099	5247	1.3530	7239	1.3796	1.020
120000	7700	5675	1.3568	7795	1.3736	1.012
130000	8300	6103	1.3600	8369	1.3713	1.008
140000	8893	6531	1.3617	8944	1.3695	1.006
150000	9442	6941	1.3603	9505	1.3694	1.007
160000	10008	7361	1.3596	10072	1.3683	1.006
170000	10565	7770	1.3597	10658	1.3717	1.009
180000	11143	8178	1.3626	11223	1.3723	1.007

We now offer a theoretical argument in favour of these asymptotic equations for all a. We will specifically carry it through for $a = 1$, but the argument is easily generalized. The case $a = 1$ is the only one which Hardy and Littlewood treated in detail. Their computation, however, was deep and function-theoretic. In contrast, the present argument is elementary, [9]. It will be assumed that the reader is acquainted with the $n^2 + 1$ sieve which is described in detail in [3].

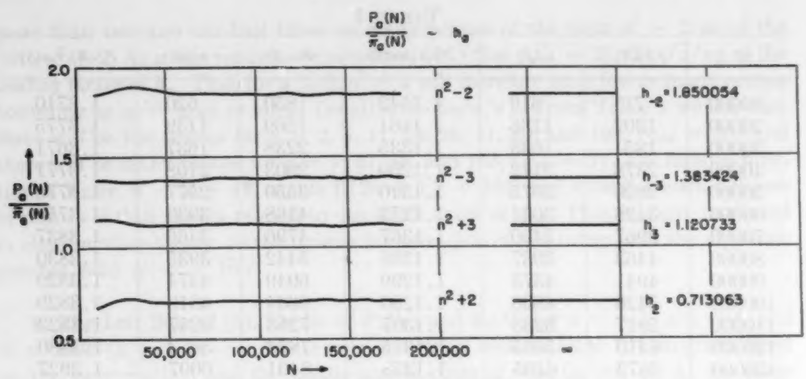

FIG. 1.—The Hardy Littlewood Conjecture.

Consider the infinite product (3) for h_1, *not* in the form in which it was given by Hardy and Littlewood, (2),

$$h_1 = \left(1 + \frac{1}{3-1}\right)\left(1 - \frac{1}{5-1}\right)\left(1 + \frac{1}{7-1}\right)\left(1 + \frac{1}{11-1}\right)\left(1 - \frac{1}{13-1}\right)\cdots,$$

since this masks its true nature; but in the equivalent form

$$h_1 = \frac{1}{\left(1-\frac{1}{3}\right)} \cdot \frac{\left(1-\frac{2}{5}\right)}{\left(1-\frac{1}{5}\right)} \cdot \frac{1}{\left(1-\frac{1}{7}\right)} \cdot \frac{1}{\left(1-\frac{1}{11}\right)} \cdot \frac{\left(1-\frac{2}{13}\right)}{\left(1-\frac{1}{13}\right)} \cdots$$

or, even better, as

$$(23)\quad h_1 = \frac{\left(1-\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)} \cdot \frac{1}{\left(1-\frac{1}{3}\right)} \cdot \frac{\left(1-\frac{2}{5}\right)}{\left(1-\frac{1}{5}\right)} \cdot \frac{1}{\left(1-\frac{1}{7}\right)} \cdot \frac{1}{\left(1-\frac{1}{11}\right)} \cdot \frac{\left(1-\frac{2}{13}\right)}{\left(1-\frac{1}{13}\right)} \cdots.$$

Now for a suitably large N let w^* be the greatest prime satisfying $w \leqq N$ and let p^* be the greatest prime of the form $4m + 1$ which satisfies $p \leqq N$. We write the corresponding *partial* product of (23), which approximates h_1, as follows:

$$(24)\qquad h_1 \approx N \cdot \frac{N\left(1-\frac{1}{2}\right)\left(1-\frac{2}{5}\right)\left(1-\frac{2}{13}\right)\cdots\left(1-\frac{2}{p^*}\right)}{N^2\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{w^*}\right)}.$$

Now this approximation to h_1 is in turn seen to be approximated (and we will inquire later as to the degree of the approximation) by N times the ratio of the primes which remain in two sieves, the Eratosthenes sieve (for all primes) from $n = 1$ to $n = N^2$ in the denominator and the $n^2 + 1$ sieve from $n^2 + 1 = 2$ to $n^2 + 1 = N^2 + 1$ in the numerator.

Without attempting precision at this point—that is, without bounding the error—we note that in the Eratosthenes sieve one first strikes out the multiples of 2. This leaves $N^2(1 - \frac{1}{2})$ numbers (with an error of 0 or $\frac{1}{2}$). One then strikes out the remaining multiples of 3 leaving $N^2(1 - \frac{1}{2})(1 - \frac{1}{3})$ numbers (again except for a possible end-effect correction.) Continuing with the primes 5, 7, $\cdots$, w^* creates the denominator of (24). The latter therefore equals

$$\pi(N^2) - \pi(N) + E(N),$$

the number of primes up to N^2 minus the number of primes up to N, with an end-effects error, $E(N)$, which is not yet bounded. We note that

$$\pi(N^2) - \pi(N) \sim \frac{N}{2}\pi(N) \sim N\bar{\pi}_1(N)$$

by the prime number theorem.

In the $n^2 + 1$ sieve we first factor a 2 from all numbers where $n = 2m + 1$ leaving $N(1 - \frac{1}{2})$ of the numbers (except for an end-effect error). We then factor a 5 where $n = 5m + 2$ and where $n = 5m + 3$. This leaves $N(1 - \frac{1}{2})(1 - \frac{2}{5})$ numbers (except for the end-effect error). Continuing with all primes of the form $4m + 1$; 13, 17, $\cdots$, p^* generates the numerator. The latter therefore equals

$$P(N) - P(\sqrt{N - 1}) + e(N),$$

the number of primes of the form $n^2 + 1$ up to $N^2 + 1$ minus the number of such primes up to N with an end effect $e(N)$.

Therefore, we *may* write

$$h_1 = \lim_{N\to\infty} \frac{P(N) - P(\sqrt{N - 1}) + e(N)}{\bar{\pi}_1(N) + E(N)/N}, \tag{25}$$

while what we would *like* to write is

$$h_1 = \lim_{N\to\infty} \frac{P(N)}{\bar{\pi}_1(N)}.$$

Now by Merten's Theorem the denominator of (24) is asymptotic to $N^2e^{-\gamma}/\log N$ where γ is Euler's constant [10]. Therefore the end effect, $E(N)/N$, is *not* negligible compared with $\bar{\pi}_1(N)$. Instead we have

$$\frac{E(N)/N}{\bar{\pi}_1(N)} \sim 0.1229 = 2e^{-\gamma} - 1. \tag{26}$$

If we could show

$$\frac{e(N)}{P(N) - P(\sqrt{N - 1})} \sim 2e^{-\gamma} - 1 \tag{27}$$

all would be well, but the difficulty of the problem is such that we cannot even prove that the left side of (27) is *bounded* from above. If we could do that, we would at least have $P(N) \to \infty$ but even this "weak" result eludes us.

It is of interest to analyze this difficulty. Let

$$D(N) = P(N) - P(\sqrt{N - 1}) \tag{28}$$

and

$$S(N) = N\left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{5}\right) \cdots \left(1 - \frac{2}{p^*}\right). \tag{29}$$

Then the conjectured relation (27) is equivalent to the conjecture

$$\frac{S(N)}{D(N)} \sim 2e^{-\gamma} = 1.1229. \tag{30}$$

Now from the sieve for $n^2 + 1$, [3], we can obtain an *exact* formula for $D(N)$ by using the "integer part of x" function, $[x]$. Consider the set of numbers obtained from

$$d = 2^a \cdot 5^b \cdot 13^c \cdots p^{*s}$$

by assigning (in all possible ways) 0 and 1 to the exponents $a, b, c, \cdots$. For each such d, let A_i be the solutions of

$$A^2 \equiv -1 \pmod d$$

which satisfy

$$0 \leqq A < d.$$

Then if d is a product of α primes, we have

$$D(N) = \sum_d (-1)^\alpha \sum_i \left[\frac{N + A_i}{d}\right]. \tag{31}$$

It may be seen that if there are M primes of the form $4m + 1$ which are $\leqq N$, then there will be $2 \cdot 3^M$ terms in this sum. Even for a very modest N, say 15, we have $p^* = 13$, $M = 2$, and there are already 18 terms. Specifically,

$$\begin{aligned} D(N) = [N] &- \left[\frac{N+1}{2}\right] - \left[\frac{N+3}{5}\right] - \left[\frac{N+2}{5}\right] + \left[\frac{N+7}{10}\right] + \left[\frac{N+3}{10}\right] \\ &- \left[\frac{N+8}{13}\right] - \left[\frac{N+5}{13}\right] + \left[\frac{N+21}{26}\right] + \left[\frac{N+5}{26}\right] + \left[\frac{N+57}{65}\right] \\ &+ \left[\frac{N+47}{65}\right] + \left[\frac{N+18}{65}\right] + \left[\frac{N+8}{65}\right] - \left[\frac{N+83}{130}\right] - \left[\frac{N+73}{130}\right] \\ &- \left[\frac{N+57}{130}\right] - \left[\frac{N+47}{130}\right]. \end{aligned}$$

In general, it is easily seen, the formula for $S(N)$ may be obtained from that for $D(N)$ by deleting the A_i and the square brackets. Thus for $N = 15$ in the example, we have

$$\begin{aligned} S(N) &= N - \frac{N}{2} - \frac{2N}{5} + \frac{2N}{10} - \frac{2N}{13} + \frac{2N}{26} + \frac{4N}{65} - \frac{4N}{130} \\ &= N\left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{5}\right)\left(1 - \frac{2}{13}\right). \end{aligned}$$

TABLE 6

N	$S(N)$	$D(N)$	$S(N)/D(N)$
100	16.261	15	1.016
200	28.252	28	1.009
300	39.800	42	0.948
400	50.696	51	0.994
500	61.344	62	0.989
600	71.763	68	1.055
700	81.656	78	1.047
800	91.345	87	1.050
900	101.075	92	1.099
1000	110.901	102	1.087
1100	119.913	112	1.071
1200	129.451	122	1.061
1300	138.223	128	1.080
1400	147.754	140	1.055
1500	156.790	150	1.045

For N small, $S(N)$ and $D(N)$ are nearly equal; e.g., $S(15) = 3.81$, $D(15) = 4$ As N increases, $S(N)$ gradually pulls ahead of $D(N)$, as is seen in Table 6.

The end effect

$$e(N) = S(N) - D(N)$$

is given by

$$e(N) = \sum_d (-1)^a \sum_i \left\{\frac{N}{d} - \left[\frac{N + A_i}{d}\right]\right\}. \tag{32}$$

Since the quantity in each brace is smaller in magnitude than unity, it is easy enough to bound $e(N)$. What is difficult to obtain is a sufficiently *good* bound—that is, to prove in general, the extensive cancellation of terms of opposite sign which occurs in the sum of (32). The essential difficulty stems from the very rapid increase in the number of terms, $2 \cdot 3^M$.

Techniques of deleting or combining terms, in sieve formulations of related problems, have been devised by Brun and others [11] but to date nothing sufficiently sharp has been developed. A general assessment of sieve techniques given by Selberg [12] is not encouraging.

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1. G. H. HARDY & J. E. LITTLEWOOD, "Partitio numerorum III: On the expression of a number as a sum of primes," *Acta Math.*, v. 44, 1923, p. 48.
2. A. E. WESTERN, "Note on the number of primes of the form $n^2 + 1$," Cambridge Phil. Soc., *Proc.*, v. 21, 1922, p. 108–109.
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6. The numbers $b(s)$ also arise in an entirely different connection—they are related to the number of distinct *circular parity switches of order s*. See DANIEL SHANKS, "A circular parity switch and applications to number theory," *Notices*, Amer. Math. Soc., v. 5, 1958, p. 96. Abstract 543-7. It was in *this* connection that the author first noted the unusual proof of a special case of the Fermat "little" theorem—see (9a) above. Likewise it was in this connection that BERNARD ELPSAS, in a private communication to the author (Sept. 3, 1958), developed the formula (9).

7. E. LANDAU, *Aus der elementaren Zahlentheorie*, Chelsea, 1946, Part IV, Chap. 6-9.

8. FLETCHER, MILLER & ROSENHEAD, *Index of Mathematical Tables*, McGraw-Hill, 1946, p. 42, 43, p. 63. The correspondence between our notation and theirs is as follows: $L_1(s) = u_n$, $L_2(s) = p_n$, $L_{-2}(s) = q_n$, $L_3(s) = h_n$, and $L_{-3}(s) = t_n$.

9. A similar sieve argument was given for the twin prime problem in CHARLES S. SUTTON, "An investigation of the average distribution of twin prime numbers," *Jn. Math. Phys.*, v. 16, 1937, p. 1-42.

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11. ERNST TROST, *Primzahlen*, Basel, 1953, Chap. IX.

12. A. SELBERG, "The general sieve method and its place in prime number theory," *Proc.*, Inter. Congress Math., Cambridge, 1950, p. 286.

On the Propagation of Errors in the Inversion of Certain Tridiagonal Matrices

By Arnold N. Lowan

Abstract. When the differential equation of heat conduction is replaced by the implicit difference analog, one is led to the solution of $Ay = \mathbf{b}$ where A is a tridiagonal matrix whose elements on the principal diagonal are $= 2 + 2r$ and whose elements off the principal diagonal are $= -r$.

The system of equations may be solved by the following algorithm:

$$\beta_k = u = r^2\beta_{k-1}^{-1}, \quad \beta_1 = u_1; \qquad \gamma_k = -r\beta^{-1}; \qquad z_k = (b_k + rz_{k-1})\beta_k^{-1}, \quad z_1 = b_1 u^{-1};$$

$$y_k = z_k - \gamma_k y_{k+1}, \qquad y_M = z_M.$$

An upper bound of the round-off errors in the computed values of the y_k's is obtained. An actual test case showed that the theoretical upper bound is about four times larger than the true round-off error. Moreover, the theoretical upper bound does not seem to vary appreciably with r.

When the differential equation of heat conduction

$$\frac{\partial T}{\partial t} = \sigma \frac{\partial^2 T}{\partial x^2}, \qquad 0 \leqq x \leqq a, \quad t > 0$$

is replaced by the "implicit" difference analog

$$\frac{T_{m,n+1} - T_{m,n}}{\Delta t} = \frac{\sigma}{2(\Delta x)^2}\,[T_{m-1,n+1} - 2T_{m,n+1} + T_{m+1,n+1} + T_{m-1,n} - 2T_{m,n} + T_{m+1,n}]$$

$$m = 1, 2, 3, \cdots M, \quad \Delta x = \frac{a}{M+1}$$

or

$$(1) \qquad (2 + 2r)T_{m,n+1} - r(T_{m-1,n+1} + T_{m+1,n+1}) = (2 - 2r)T_{m,n} + r(T_{m-1,n} + T_{m+1,n})$$

where $T_{m,n} = T(m\Delta x, n\Delta t)$ and $r = \sigma\Delta t/(\Delta x)^2$ it is a known fact that the difference scheme (1) is unconditionally stable [1]. If the desired solution is required to vanish on the boundaries $x = 0$ and $x = a$, the system of equations (1) may be written in the compact form†

$$(1^*) \qquad A\mathbf{T}_{n+1} = B\mathbf{T}_n = \mathbf{b} \quad \text{(say)}$$

where A is a tridiagonal matrix whose elements on the principal diagonal are $= 2 + 2r$ while the elements off the principal diagonal are $= -r$.

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† When the temperature is prescribed on the boundaries, equation (1*) is essentially unchanged except for the fact that the first and last components of **b** are slightly altered.

The system of equations (1) may be easily solved by the following algorithm [2]

(2) $$\beta_k = u - \frac{r^2}{\beta_{k-1}} \qquad k = 1, 2, 3, \cdots, M; \quad \beta_1 = u$$

(3) $$\gamma_k = -\frac{r}{\beta_k} \qquad k = 1, 2, 3, \cdots, M;$$

(4) $$z_k = \frac{1}{\beta_k}(b_k + rz_{k-1}) \qquad k = 1, 2, 3, \cdots, M; \quad z_1 = \frac{b_1}{u}$$

(5) $$y_k = z_k - \gamma_k y_{k+1} \qquad k = 1, 2, 3, \cdots, M; \quad y_M = z_M$$

where we have written u for $2 + 2r$ and we have denoted the components of $\mathbf{T}_{n+1}$ by y_k. The question arises: if the computations involved in the above algorithm are carried to p decimals (i.e., if products and ratios are rounded to p decimals) what is the upper bound of the round-off errors in the computed values of y_k?

In the derivation of the desired upper bound we shall require a lower bound of the β_k's and upper bounds of γ_k, z_k and y_k. If in (2) we put $k = 2, 3, \cdots$ we get

(6) $$\beta_2 = u - \frac{r^2}{\beta_1} = \beta_1 - \frac{r^2}{\beta_1}$$
$$\beta_3 = u - \frac{r^2}{\beta_2}$$
$$\vdots$$

From the first of the above equations it is clear that $\beta_2 < \beta_1$. From the first two equations it follows that

$$\beta_3 - \beta_2 = r^2\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) < 0.$$

Thus $\beta_3 < \beta_2$. Similarly it may be shown that $\beta_4 < \beta_3, \cdots, \beta_k < \beta_{k-1}$. Thus the β_k's form a monotonically decreasing sequence. It may be readily shown that the lower limit of the sequence, to be denoted by β_*, is the larger of the two roots of the quadratic equation

(7) $$x^2 - (2 + 2r)x + r^2 = 0.$$

Accordingly

(8) $$\beta_* = 1 + r + \sqrt{1 + 2r}.$$

From (3) it follows that $|\gamma_k| < r/\beta_k$. If then γ^* denotes an upper bound of $|\gamma_k|$ we may put

(9) $$\gamma^* = \frac{r}{\beta_*} = \frac{r}{1 + r + \sqrt{1 + 2r}}.$$

If in (4) we put $k = 2, 3, \cdots$ and subsequently eliminate $z_2, z_3, \cdots, z_{k-1}$ we ultimately get

$$z_k = \frac{b_k}{\beta_k} + \frac{rb_{k-1}}{\beta_k\beta_{k-1}} + \frac{r^2 b_{k-2}}{\beta_k\beta_{k-1}\beta_{k-2}} + \cdots \frac{r^{k-1} b_1}{\beta_k\beta_{k-1}\cdots\beta_1}$$

whence

$$|z_k| \leqq \frac{b^*}{\beta_*}\left[1 + \frac{r}{\beta_*} + \frac{r^2}{\beta_*^2} + \cdots \left(\frac{r}{\beta_*}\right)^{k-1}\right] \cong \frac{b^*}{\beta_*} \cdot \frac{1}{1 - \frac{r}{\beta_*}} = \frac{b^*}{\beta_* - r}$$

where b^* is the largest of the absolute values of b_k. If then z^* denotes an upper bound of $|z_k|$ we may put

$$z^* = \frac{b^*}{\beta_* - r}. \tag{10}$$

Finally from (5) we readily get

$$\begin{aligned}
y_M &= z_M \\
y_{M-1} &= z_{M-1} - \gamma_{M-1} z_M \\
y_{M-2} &= z_{M-2} - \gamma_{M-2} z_{M-1} + \gamma_{M-2}\gamma_{M-1} z_M \\
&\vdots \\
y_1 &= z_1 - \gamma_1 z_2 + \gamma_1\gamma_2 z_3 - \cdots (-1)^{M-1}\gamma_1\gamma_2 \cdots \gamma_{M-1} z_1.
\end{aligned}$$

From the above system of equations it is clear that

$$\begin{aligned}
y^* &= z^*(1 + \gamma^* + \gamma^{*2} \cdots + \gamma^{*M-1}) \\
&\cong \frac{z^*}{1 - \gamma^*} = \frac{b^*}{\beta_* - r} \cdot \frac{1}{r - \frac{r}{\beta_*}} = \frac{b^*\beta_*}{(\beta_* - r)^2}
\end{aligned} \tag{11}$$

is an upper bound of the absolute values of the y_k's.

We now turn to the evaluation of upper bounds of the errors in the β_k's, γ_k's, z_k's and y_k's. It will be convenient to denote by $E(\beta_k)$ the absolute value of the error in β_k and by $E^*(\beta)$ an upper bound of the errors in the β_k's. A similar notation will be used for the γ_k's, z_k's and y_k's. From (2) we have

$$E(\beta_2) = \frac{r^2}{\beta_1^2} E(\beta_1) + \delta \leqq \frac{r^2}{\beta_*^2} E(\beta_1) + \delta$$

where $\delta = \frac{1}{2} \times 10^{-p}$ is the maximum round-off error. Similarly

$$\begin{aligned}
E(\beta_3) &= \frac{r^2}{\beta_2^2} E(\beta_2) + \delta \leqq \frac{r^2}{\beta_*^2} E(\beta_2) + \delta \\
&= \frac{r^2}{\beta_*^2}\left[\frac{r^2}{\beta_*^2} E(\beta_1) + \delta\right] + \delta \\
&= \left(1 + \frac{r^2}{\beta_*^2}\right)\delta + \left(\frac{r^2}{\beta_*^2}\right)^2 E(\beta_1).
\end{aligned}$$

Proceeding in this manner we ultimately get

$$\begin{aligned}
E(\beta_n) &\leqq \left[1 + \left(\frac{r^2}{\beta_*^2}\right) + \cdots \left(\frac{r^2}{\beta_*^2}\right)^{M-2}\right]\delta + \left(\frac{r^2}{\beta_*^2}\right)^{M-1} E(\beta_1) \\
&\cong \frac{1}{1 - \frac{r^2}{\beta_*^2}}\,\delta = \frac{\beta_*^2}{\beta_*^2 - r^2}\,\delta
\end{aligned}$$

where we have neglected the second term of the above inequality since $r < \beta_*$. Thus

$$E^*(\beta) = \frac{\beta_*^2}{\beta_*^2 - r^2}\,\delta \tag{12}$$

is an upper bound of the absolute values of the $E(\beta_k)$'s.

Consider now the evaluation of $E^*(\gamma)$. From (3) it follows that

$$E(\gamma_k) = \frac{r}{\beta_k^2} E(\beta_k) + \delta$$

$$< \frac{r}{\beta_*^2} E^*(\beta) + \delta$$

whence

$$\begin{aligned} E^*(\gamma) &= \frac{r}{\beta_*^2} E^*(\beta) + \delta = \frac{r}{\beta_*^2} \cdot \frac{\beta_*^2}{\beta_*^2 - r^2}\,\delta + \delta \\ &= \left(1 + \frac{r}{\beta_*^2 - r^2}\right)\delta. \end{aligned} \tag{13}$$

Consider next the evaluation of $E^*(z)$. From (4) we get:

$$\begin{aligned} E(z_k) &= \frac{1}{\beta_k^2}\{(b_k + rz_{k-1})E(\beta_k) + \beta_k[E(b_k) + rE(z_{k-1})]\} + \delta \\ &\leqq \frac{1}{\beta_*^2}(b^* + rz^*)E^*(\beta) + \frac{1}{\beta_*}E^*(b) + \frac{r}{\beta_*}E(z_{k-1}) + \delta \\ &= \frac{1}{\beta_*^2}(b^* + rz^*)\frac{\beta_*^2}{\beta_*^2 - r^2} + \frac{1}{\beta_*}E^*(b) + \frac{r}{\beta_*}E(z_{k-1}) + \delta \\ &= \left(1 + \frac{b^* + rz^*}{\beta_*^2 - r^2}\right)\delta + \frac{1}{\beta_*}E^*(b) + \frac{r}{\beta_*}E(z_{k-1}). \end{aligned}$$

Proceeding as in the evaluation of $E^*(\beta)$ we ultimately get

$$E^*(z) = \frac{\beta_*}{\beta_* - r}\left(1 + \frac{b^* + rz^*}{\beta_*^2 - r^2}\right)\delta + \frac{E^*(b)}{\beta_* - r}. \tag{14}$$

If in the last equation we replace z^* by its expression from (10) we ultimately get

$$E^*(z) = \frac{\beta_*}{\beta_* - r}\left[1 + \frac{b^*\beta_*}{(\beta_* - r)(\beta_*^2 - r^2)}\right] + \frac{E^*(b)}{\beta_* - r}. \tag{15}$$

If on the other hand we replace z^* in (14) by Z, the largest absolute value of the z_k's we obtain

$$E^*(z) = \frac{\beta_*}{\beta_* - r}\left(1 + \frac{b^* + rZ}{\beta_*^2 - r^2}\right)\delta + \frac{E^*(b)}{\beta_* - r}. \tag{15*}$$

While the expression in (15) is an upper bound of the errors in the z_k's, it is reasonable to refer to the expression in (15*) as the least upper bound of the errors in the z_k's.

Finally, consider the evaluation of $E^*(y)$. From (5) we get

$$E(y_k) = E(z_k) + \gamma_k E(y_{k+1}) + y_{k+1}E(\gamma_k) + \delta$$

whence

$$E(y_k) \leqq E^*(z) + \gamma^* E(y_{k+1}) + y^* E^*(\gamma) + \delta. \tag{16}$$

Substituting for $E^*(z)$, γ^*, y^* and $E^*(\gamma)$ their expressions from (15), (9), (11), and (13) the last inequality becomes

$$\begin{aligned} E(y_k) \leqq {} & \frac{\beta_*}{\beta_* - r}\left[1 + \frac{b^*\beta_*}{(\beta_* - r)(\beta_*^2 - r^2)}\right]\delta + \frac{E^*(b)}{\beta_* - r} \\ & + \frac{b^*\beta_*}{(\beta_* - r)^2}\left(1 + \frac{r}{\beta_*^2 - r^2}\right)\delta + \delta + \frac{r}{\beta_*}E(y_{k+1}). \end{aligned} \tag{17}$$

Proceeding again as in the evaluation of $E^*(\beta)$, the last inequality ultimately yields:

$$\begin{aligned} E^*(y) = {} & \frac{\beta_*}{\beta_* - r}\left\{\frac{\beta_*}{\beta_* - r}\left[1 + \frac{b^*\beta_*}{(\beta_* - r)(\beta_*^2 - r^2)}\right]\right. \\ & \left. + \frac{b^*\beta_*}{(\beta_* - r)^2}\left(1 + \frac{r}{(\beta_*^2 - r^2)}\right) + 1\right\}\delta + \frac{\beta_*}{(\beta_* - r)^2}E^*(b). \end{aligned} \tag{18}$$

If, on the other hand, we substitute for $E^*(z)$ in (16) its expression from (15*) and replace y^* by Y the largest of the absolute values of the y_k's, while γ^* and $E^*(\gamma)$ are replaced by their expressions from (9) and (13), we obtain as the counterpart of (17)

$$\begin{aligned} E(y_k) \leqq {} & \frac{\beta_*}{\beta_* - r}\left[1 + \frac{b^* + rZ}{\beta_*^2 - r^2}\right]\delta + \frac{E^*(b)}{\beta_* - r} \\ & + Y\left(1 + \frac{r}{\beta_*^2 - r^2}\right)\delta + \delta + \frac{r}{\beta_*}E(y_{k+1}). \end{aligned} \tag{17*}$$

Proceeding again as in the evaluation of $E^*(\beta)$, the last inequality ultimately yields:

$$\begin{aligned} E^*(y) = {} & \frac{\beta_*}{\beta_* - r}\left\{\frac{\beta_*}{\beta_* - r}\left[1 + \frac{b^* + rZ}{\beta_*^2 - r^2}\right] + Y\left(1 + \frac{r}{\beta_*^2 - r^2}\right) + 1\right\}\delta \\ & + \frac{\beta_*}{(\beta_* - r)^2}E^*(b). \end{aligned} \tag{18*}$$

It will be convenient to rewrite the last equation in the form

$$E^*(y) = S_0(r) + b^*S_1(r) + YS_2(r) + ZS_3(r) + E^*(b)S_4(r) \tag{19}$$

where

$$\begin{cases} S_0(r) = \dfrac{\beta_*}{\beta_* - r} + \dfrac{\beta_*^2}{(\beta_* - r)^2} \\[2ex] S_1(r) = \dfrac{\beta_*^2}{(\beta_* - r)^2(\beta_*^2 - r^2)} \\[2ex] S_2(r) = \dfrac{\beta_*}{\beta_* - r}\left(1 + \dfrac{r}{\beta_*^2 - r^2}\right) \\[2ex] S_3(r) = \dfrac{r\beta_*^2}{(\beta_* - r)^2(\beta_*^2 - r^2)} = rS_1(r) \\[2ex] S_4(r) = \dfrac{\beta_*}{(\beta_* - r)^2}. \end{cases} \tag{20}$$

In (19) in conjunction with (20) we have an upper bound of the round-off errors in the values of the y_k's—the solutions of $A\mathbf{y} = \mathbf{b}$. In case the b_k's are exact we must of course put $E^*(b) = 0$.

To test the formula (18*) the exact components of $\mathbf{b}$ were computed from $A\mathbf{y} = \mathbf{b}$ where A is a 20×20 tridiagonal matrix of the type above considered with $r = 1$ and $r = 2$ and the 20 components of $\mathbf{y}$ were arbitrarily assigned; the values of the components were then calculated by the above algorithm in terms of the exact values of b_k. The computations were carried to eight decimals. The maximum discrepancy between the exact values of the y_k's and the corresponding computed values was two units in the last place. The upper bound of the round-off errors evaluated from (19) in conjunction with (20) was eight units in the last place in the case $r = 1$ and seven units in the last place in the case $r = 2$. The estimated upper bounds of the round-off errors must be considered as indeed very close to the actual round-off errors.

The writer wishes to express his appreciation to Roy Steeves who carried out the above-mentioned test.

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1. The implicit scheme was first suggested by J. Crank and P. Nicolson in the paper entitled, "A practical method for the numerical evaluation of solutions of differential equations of the heat conduction type," Camb. Phil. Soc. *Proc.*, v. 43, 1943, p. 50–67. Its stability is discussed by G. G. O'Brien, Morton A. Hyman, and Sidney Kaplan in "A Study of the numerical solution of partial differential equations," *Jn. Math. and Phys.*, v. 29, 1950/51, p. 223–251. It is also discussed in the writer's book *The Operator Approach to Problems of Stability and Convergence*, Scripta Mathematica, 1957.

2. See for instance M. Lotkin's "The numerical integration of heat conduction equations," *Jn. Math. and Phys.*, v. 37, 1958, p. 178, and R. D. Richtmyer, *Difference Methods for Initial Value Problems*, Chap. VI., Interscience, N. Y., 1957.

Tabulation of Coefficients for Operations on Taylor Series

By Daniel C. Fielder

1. Introduction. When certain operations are performed on polynomials or power series, the results often assume the form of a series of ascendingly indexed terms. A typical term, the kth, appears as the product of two distinct types of factors. One of these factors is a multiplier peculiar to the particular operation. The other factor is a group of literal coefficients which forms a kth order symmetric function of weight k and degree h in those literal coefficients. A detailed examination reveals that the second factor is composed of sums of combinations of appropriately subscripted and exponentiated literal coefficients. This second factor, which will be called a *literal group*, has a form common to many operations on polynomials and power series. The multiplier of each combination within a group is observed to be either a positive integer or a positive common fraction.

As is well known, the *weight* of a term in a symmetric function is the sum of the products of corresponding subscripts and exponents. The *degree* is the sum of the exponents of a product combination of literal coefficients. As an example, for $k = 6$ and $h = 3$ the kth term might be $\{6\}(g_1^2 g_4 + [2]g_1 g_2 g_3 + [\frac{1}{3}]g_2^3)$. The weight of a product combination of literal coefficients, say the $g_1^2 g_4$ term, is $1 \times 2 + 4 \times 1 = 6$, and the degree of a product combination, again using the $g_1^2 g_4$ term, is $2 + 1 = 3$. For uniformity, the identification of enclosing symbols with algebraic terms used in the above example is used throughout the paper.

Several direct and inverse operations are investigated, and methods for finding general terms for groups and multipliers are discussed. For convenience, literal groups are tabulated. Because of the common properties of the literal groups, a single table of literal groups suffices for all operations described herein.

2. Power Series of Power Series. The first operation to be considered is

$$(b_1 y + b_2 y^2 + b_3 y^3 + \cdots) = (m_1 x + m_2 x^2 + m_3 x + \cdots), \tag{1}$$

in which

$$y = g_1 x + g_2 x^2 + g_3 x^3 + \cdots. \tag{2}$$

This is essentially 6.362 of Adams [1].

A direct calculation of the first few terms leads to

$$\begin{aligned} m_1 &= \{b_1\}(g_1), \\ m_2 &= \{b_2\}(g_1^2) + \{b_1\}(g_2), \\ m_3 &= \{b_3\}(g_1^3) + \{2b_2\}(g_1 g_2) + \{b_1\}(g_3), \\ m_4 &= \{b_4\}(g_1^4) + \{3b_3\}(g_1^2 g_2) \\ &\quad + \{2b_2\}(g_1 g_3 + [\tfrac{1}{2}]g_2^2) + \{b_1\}(g_4), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,. \end{aligned} \tag{3}$$

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Table 1 presents a general form for the literal and numerical (braced) coefficients up through index 12. For any $\{t_{hk}\}$ of Table 1 except $\{t_{kk}\}$, the numerical coefficients for the operation described by (1) and (2) are hb_h. For $\{t_{kk}\}$, the numerical coefficients are b_k.

TABLE 1. *Values of literal groups, $k = 1(1)12$*

$$
\begin{aligned}
m_1 &= \{t_{11}\}(g_1),\\
m_2 &= \{t_{22}\}(g_1^2) + \{t_{21}\}(g_2),\\
m_3 &= \{t_{33}\}(g_1^3) + \{t_{32}\}(g_1g_2) + \{t_{31}\}(g_3),\\
m_4 &= \{t_{44}\}(g_1^4) + \{t_{43}\}(g_1^2g_2) + \{t_{42}\}(g_1g_3 + [\tfrac{1}{2}]g_2^2) + \{t_{41}\}(g_4),\\
m_5 &= \{t_{55}\}(g_1^5) + \{t_{54}\}(g_1^3g_2) + \{t_{53}\}(g_1^2g_3 + g_1g_2^2)\\
&\quad + \{t_{52}\}(g_1g_4 + g_2g_3) + \{t_{51}\}(g_5),\\
m_6 &= \{t_{66}\}(g_1^6) + \{t_{65}\}(g_1^4g_2) + \{t_{64}\}(g_1^3g_3 + [\tfrac{3}{2}]g_1^2g_2^2)\\
&\quad + \{t_{63}\}(g_1^2g_4 + [2]g_1g_2g_3 + [\tfrac{1}{3}]g_2^3)\\
&\quad + \{t_{62}\}(g_1g_5 + g_2g_4 + [\tfrac{1}{2}]g_3^2) + \{t_{61}\}(g_6),\\
m_7 &= \{t_{77}\}(g_1^7) + \{t_{76}\}(g_1^5g_2) + \{t_{75}\}(g_1^4g_3 + [2]g_1^3g_2^2)\\
&\quad + \{t_{74}\}(g_1^3g_4 + [3]g_1^2g_2g_3 + g_1g_2^3)\\
&\quad + \{t_{73}\}(g_1^2g_5 + [2]g_1g_2g_4 + g_2^2g_3 + g_1g_3^2)\\
&\quad + \{t_{72}\}(g_1g_6 + g_2g_5 + g_3g_4) + \{t_{71}\}(g_7),\\
m_8 &= \{t_{88}\}(g_1^8) + \{t_{87}\}(g_1^6g_2) + \{t_{86}\}(g_1^5g_3 + [\tfrac{5}{2}]g_1^4g_2^2)\\
&\quad + \{t_{85}\}(g_1^4g_4 + [4]g_1^3g_2g_3 + [2]g_1^2g_2^3)\\
&\quad + \{t_{84}\}(g_1^3g_5 + [3]g_1^2g_2g_4 + [\tfrac{3}{2}]g_1^2g_3^2 + [3]g_1g_2^2g_3 + [\tfrac{1}{4}]g_2^4)\\
&\quad + \{t_{83}\}(g_1^2g_6 + [2]g_1g_2g_5 + [2]g_1g_3g_4 + g_2^2g_4 + g_2g_3^2)\\
&\quad + \{t_{82}\}(g_1g_7 + g_2g_6 + g_3g_5 + [\tfrac{1}{2}]g_4^2) + \{t_{81}\}(g_8),\\
m_9 &= \{t_{99}\}(g_1^9) + \{t_{98}\}(g_1^7g_2) + \{t_{97}\}(g_1^6g_3 + [3]g_1^5g_2^2)\\
&\quad + \{t_{96}\}(g_1^5g_4 + [5]g_1^4g_2g_3 + [\tfrac{10}{3}]g_1^3g_2^3)\\
&\quad + \{t_{95}\}(g_1^4g_5 + [4]g_1^3g_2g_4 + [2]g_1^3g_3^2 + [6]g_1^2g_2^2g_3 + g_1g_2^4)\\
&\quad + \{t_{94}\}(g_1^3g_6 + [3]g_1^2g_2g_5 + [3]g_1^2g_3g_4 + [3]g_1g_2g_3^2 + [3]g_1g_2^2g_4 + g_2^3g_3)\\
&\quad + \{t_{93}\}(g_1^2g_7 + [2]g_1g_2g_6 + [2]g_1g_3g_5 + g_1g_4^2 + g_2^2g_5 + [2]g_2g_3g_4 + [\tfrac{1}{3}]g_3^3)\\
&\quad + \{t_{92}\}(g_1g_8 + g_2g_7 + g_3g_6 + g_4g_5) + \{t_{91}\}(g_9),\\
m_{10} &= \{t_{10,10}\}(g_1^{10}) + \{t_{10,9}\}(g_1^8g_2) + \{t_{10,8}\}(g_1^7g_3 + [\tfrac{7}{2}]g_1^6g_2^2)\\
&\quad + \{t_{10,7}\}(g_1^6g_4 + [6]g_1^5g_2g_3 + [5]g_1^4g_2^3) + \{t_{10,6}\}(g_1^5g_5 + [5]g_1^4g_2g_4\\
&\quad + [\tfrac{5}{2}]g_1^4g_3^2 + [10]g_1^3g_2^2g_3 + [\tfrac{5}{2}]g_1^2g_2^4) + \{t_{10,5}\}(g_1^4g_6 + [4]g_1^3g_2g_5\\
&\quad + [4]g_1^3g_3g_4 + [6]g_1^2g_2^2g_4 + [6]g_1^2g_3^2g_2 + [4]g_2^3g_3g_1 + [\tfrac{1}{5}]g_2^5)\\
&\quad + \{t_{10,4}\}(g_1^3g_7 + [3]g_1^2g_2g_6 + [3]g_1^2g_3g_5 + [\tfrac{3}{2}]g_1^2g_4^2 + [3]g_1g_2^2g_5\\
&\quad + [6]g_1g_2g_2g_4 + g_2^3g_4 + [\tfrac{3}{2}]g_2^2g_3^2 + g_1g_3^3) + \{t_{10,3}\}(g_1^2g_8\\
&\quad + [2]g_1g_2g_7 + [2]g_1g_3g_6 + [2]g_1g_4g_5 + [2]g_2g_3g_5 + g_2g_4^2 + g_2^2g_6\\
&\quad + g_3^2g_4) + \{t_{10,2}\}(g_1g_9 + g_2g_8 + g_3g_7 + g_4g_6 + [\tfrac{1}{2}]g_5^2) + \{t_{10,1}\}(g_{10}),\\
m_{11} &= \{t_{11,11}\}(g_1^{11}) + \{t_{11,10}\}(g_1^9g_2) + \{t_{11,9}\}(g_1^8g_3 + [4]g_1^7g_2^2)\\
&\quad + \{t_{11,8}\}(g_1^7g_4 + [7]g_1^6g_2g_3 + [7]g_1^5g_2^3) + \{t_{11,7}\}(g_1^6g_5 + [6]g_1^5g_2g_4\\
&\quad + [3]g_1^5g_3^2 + [15]g_1^4g_2^2g_3 + [5]g_1^3g_2^4) + \{t_{11,6}\}(g_1^5g_6 + [5]g_1^4g_2g_5\\
&\quad + [5]g_1^4g_3g_4 + [10]g_1^2g_2^3g_3 + [10]g_1^3g_2^2g_4 + [10]g_1^3g_2g_3^2 + g_1g_2^5)\\
&\quad + \{t_{11,5}\}(g_1^4g_7 + [4]g_1^3g_2g_6 + [4]g_1^3g_3g_5 + [2]g_1^3g_4^2 + [6]g_1^2g_2^2g_5\\
&\quad + [12]g_1^2g_2g_3g_4 + [4]g_1g_2^3g_4 + [6]g_1g_2^2g_3^2 + g_2^4g_3 + [2]g_1^2g_3^3)\\
&\quad + \{t_{11,4}\}(g_1^3g_8 + [3]g_1^2g_2g_7 + [3]g_1^2g_3g_6 + [3]g_1^2g_4g_5 + [3]g_1g_2^2g_6\\
&\quad + [6]g_1g_2g_3g_5 + [3]g_1g_2g_4^2 + [3]g_1g_3^2g_4 + g_2^3g_5 + [3]g_2^2g_3g_4\\
&\quad + g_3^3g_2) + \{t_{11,3}\}(g_1^2g_9 + [2]g_1g_2g_8 + [2]g_1g_3g_7 + [2]g_1g_4g_6\\
&\quad + g_1g_5^2 + g_2^2g_7 + [2]g_2g_3g_6 + [2]g_2g_4g_5 + g_3g_4^2 + g_3^2g_5)\\
&\quad + \{t_{11,2}\}(g_1g_{10} + g_2g_9 + g_3g_8 + g_4g_7 + g_5g_6) + \{t_{11,1}\}(g_{11}),
\end{aligned}
$$

TABLE 1—*Continued*

$$
\begin{aligned}
m_{12} = {} & \{t_{12,12}\}(g_1^{12}) + \{t_{12,11}\}(g_1^{10}g_2) + \{t_{12,10}\}(g_1^9g_3 + [\tfrac{9}{2}]g_1^8g_2^2) \\
& + \{t_{12,9}\}(g_1^8g_4 + [8]g_1^7g_2g_3 + [\tfrac{28}{3}]g_1^6g_2^3) + \{t_{12,8}\}(g_1^7g_5 + [\tfrac{7}{2}]g_1^6g_3^2 \\
& + [7]g_1^6g_2g_4 + [\tfrac{35}{4}]g_1^4g_2^4 + [21]g_1^5g_2^2g_3) + \{t_{12,7}\}(g_1^6g_6 \\
& + [3]g_1^2g_2^5 + [6]g_1^5g_2g_5 + [6]g_1^5g_3g_4 + [15]g_1^4g_2^2g_4 + [15]g_1^4g_2g_3^2 \\
& + [20]g_1^3g_2^3g_3) + \{t_{12,6}\}(g_1^5g_7 + [\tfrac{5}{2}]g_1^4g_4^2 + [\tfrac{10}{3}]g_1^3g_3^3 + [5]g_1^4g_2g_6 \\
& + [5]g_1^4g_3g_5 + [5]g_1g_2^4g_3 + [10]g_1^3g_2^2g_5 + [10]g_1^2g_2^3g_4 \\
& + [15]g_1^2g_2^2g_3^2 + [20]g_1^3g_2g_3g_4 + [\tfrac{1}{6}]g_2^6) + \{t_{12,5}\}(g_1^4g_8 + g_2^4g_4 \\
& + [2]g_2^3g_3^2 + [4]g_1^3g_2g_7 + [4]g_1^3g_3g_6 + [4]g_1^3g_4g_5 + [4]g_1g_2^3g_5 \\
& + [4]g_1g_2g_3^3 + [6]g_1^2g_2^2g_6 + [6]g_1^2g_2g_4^2 + [6]g_1^2g_3^2g_4 \\
& + [12]g_1^2g_2g_3g_5 + [12]g_1g_2^2g_3g_4) + \{t_{12,4}\}(g_1^3g_9 + g_2^3g_6 \\
& + [\tfrac{3}{2}]g_1^2g_5^2 + [\tfrac{3}{2}]g_2^2g_4^2 + [3]g_1^2g_2g_8 + [3]g_1^2g_3g_7 + [3]g_1g_2^2g_7 \\
& + [3]g_1^2g_4g_6 + [3]g_1g_3^2g_5 + [3]g_1g_3g_4^2 + [3]g_2^2g_3g_5 \\
& + [3]g_2g_3^2g_4 + [6]g_1g_2g_3g_6 + [6]g_1g_2g_4g_5 + [\tfrac{1}{4}]g_3^4) \\
& + \{t_{12,3}\}(g_1^2g_{10} + g_2^2g_8 + g_2g_5^2 + g_3^2g_6 + [2]g_1g_2g_9 + [2]g_1g_3g_8 \\
& + [2]g_1g_4g_7 + [2]g_1g_5g_6 + [2]g_2g_3g_7 + [2]g_2g_4g_6 + [2]g_3g_4g_5 \\
& + [\tfrac{1}{3}]g_4^3) + \{t_{12,2}\}(g_1g_{11} + g_2g_{10} + g_3g_9 + g_4g_8 + g_5g_7 + [\tfrac{1}{2}]g_6^2) \\
& + \{t_{12,1}\}(g_{12}).
\end{aligned}
$$

3. Reversion of Power Series. (See 6.26 of Adams [1] and Van Orstrand [2].) If

$$z = x - g_1x^2 - g_2x^3 - g_3x^4 - \cdots \tag{4}$$

is reversed to yield

$$x = z + m_1z^2 + m_2z^3 + m_3z^4 + \cdots, \tag{5}$$

the result is similar to that noted in the preceding section. The first few terms are

$$
\begin{aligned}
m_1 &= \{1\}(g_1), \\
m_2 &= \{2\}(g_1^2) + \{1\}(g_2), \\
m_3 &= \{5\}(g_1^3) + \{5\}(g_1g_2) + \{1\}(g_3), \\
m_4 &= \{14\}(g_1^4) + \{21\}(g_1^2g_2) \\
&\quad + \{6\}(g_1g_3 + [\tfrac{1}{2}]g_2^2) + \{1\}(g_4), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned}
\tag{6}
$$

Table 1 applies as before. For any $\{t_{kh}\}$ of Table 1 except $\{t_{kk}\}$, the numerical coefficients for the operation described by (4) and (5) are the familiar binomial coefficients $\binom{k+h}{k+1}$. For $\{t_{kk}\}$, the numerical coefficients are $\left[\binom{2k}{k} - \binom{2k}{k-1}\right]$.

4. Development of a Power Series Ratio. The four succeeding operations are based on a tabulation derived from the ratio of two power series. A short discussion of this tabulation follows. The two power series and the quotient $P(z)$ are

$$\frac{\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots}{\beta_0 + \beta_1 z + \beta_2 z^2 + \cdots} = P(z) = C_0 + C_1 z + C_2 z^2 + \cdots . \tag{7}$$

If the indicated division is performed, a triangular tabulation array can be obtained as shown below.

$$\begin{aligned} &C_0\beta_0 - \alpha_0 = 0, \\ &C_1\beta_0 + C_0\beta_1 - \alpha_1 = 0, \\ &C_2\beta_0 + C_1\beta_1 + C_0\beta_2 - \alpha_2 = 0, \\ &C_3\beta_0 + C_2\beta_1 + C_1\beta_2 + C_0\beta_3 - \alpha_3 = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned} \tag{8}$$

Thus, if the α's and β's of the array (8) are known, the C's can be found by the division indicated in (7).

5. Direct and Inverse Summation Formulas of Power of Roots. In a previous paper, the author [3] discusses means of finding sums of powers of roots of the integral rational function

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \tag{9}$$

by elaborating on a method credited to Newton [1]. Newton's method utilizes the tabulation,

$$\begin{aligned} &1a_1 + S_1a_0 = 0, \\ &2a_2 + S_1a_1 + S_2a_0 = 0, \\ &3a_3 + S_1a_2 + S_2a_1 + S_3a_0 = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \end{aligned} \tag{10}$$

where the sum of the bth powers of the roots of $f(x)$ are

$$S_b = r_1^b + r_2^b + r_3^b + \cdots . \tag{11}$$

A comparison of (8) and (10) suggests that it should be possible to find a $P(z)$ whose coefficients are functions of the a's of (10) and are simply related to S_k's. Such is indeed the case for the determination of the *direct* summation formulas. For convenience, (10) is rearranged and restated as

$$\begin{aligned} &m_1 + g_1 = 0, \\ &2m_2 + m_1g_1 + 2g_2 = 0, \\ &3m_3 + 2m_2g_1 + m_1g_2 + 3g_3 = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \end{aligned} \tag{12}$$

where $g_k = (a_k/a_0)$ and $m_k = (S_k/k)$.

For the direct summation, it is seen that $\beta_0 = 1$, $C_k = (k + 1)m_{k+1}$, $\beta_k = g_k$, and $\alpha_k = -kg_k$. Thus, $P(z)$ becomes

$$P(z) = \frac{-g_1 - 2g_1z - 3g_3z^2 - 4g_4z^3 \cdots}{1 + g_1z + g_2z^2 + g_3z^3 + \cdots} . \tag{13}$$

The first few terms of the indicated division are

$$\begin{aligned} &- g_1 + (g_1^2 - 2g_2)z + (- g_1^3 + 3g_1g_2 - 3g_3)z^2 \\ &+ (g_1^4 - 4g_1^2g_2 + 4g_1g_3 + 2g_2^2 - 4g_4)z^3 + \cdots , \end{aligned} \tag{14}$$

where $-g_1 = C_0$, $g_1^2 - 2g_2 = C_1$, etc. Thus,

$$
\begin{aligned}
m_1 &= \{-1/1\}(g_1), \\
m_2 &= \{1/2\}(g_1^2) + \{-1\}(g_2), \\
m_3 &= \{-1/3\}(g_1^3) + \{1\}(g_1g_2) + \{-1\}(g_3), \\
m_4 &= \{1/4\}(g_1^4) + \{-1\}(g_1^2g_2) \\
&\quad + \{1\}(g_1g_3 + [1/2]g_2^2) + \{-1\}(g_4), \\
&\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned} \tag{15}
$$

It is apparent that (13) not only yields the desired summations but serves as a device for generating the literal groups of Table 1. For any $\{t_{kl}\}$ of Table 1 except $\{t_{kk}\}$, the numerical coefficients for the direct summation are $(-1)^h$. For $\{t_{kk}\}$, the numerical coefficients are $(-1)^k/k$.

It is not possible to use (12) directly to find a $P(z)$ whose coefficients lead to the inverse summation forms. However, a two-step solution leading to the inverse summation forms can be found by rearranging (12) as

$$
\begin{aligned}
g_1 + B_1 &= 0, \\
g_2 + m_1g_1 + 2B_2 &= 0, \\
g_3 + m_1g_2 + 2m_2g_1 + 3B_3 &= 0, \\
&\cdots\cdots\cdots\cdots ,
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
1B_1 &= 1m_1 + 0g_1, \\
2B_2 &= 2m_2 + 1g_2, \\
3B_3 &= 3m_3 + 2g_3, \\
&\cdots\cdots\cdots .
\end{aligned} \tag{17}
$$

The relationships $g_k = (a_k/a_0)$ and $m_k = (S_k/k)$ apply as before.

For the inverse summation, it is seen that $\beta_0 = 1$, $C_k = g_k$, $\beta_k = km_k$, and $\alpha_k = (k + 1)B_{k+1}$. Thus, the $P(z)$ becomes

$$
\begin{aligned}
P(z) &= \frac{-B_1 - 2B_2z - 3B_3z^2 - 4B_4z^3 - \cdots}{1 + m_1z + 2m_2z^2 + 3m_3z^3 + \cdots} \\
&= g_1 + g_2z + g_3z^2 + g_4z^3 + \cdots .
\end{aligned} \tag{18}
$$

The first few terms are

$$
-B_1 + (B_1m_1 - 2B_2)z + (-B_1m_1^2 + 2B_2m_1 + 2B_1m_2 - 3B_3)z^2 + \cdots . \tag{19}
$$

Through use of (17), the B terms are eliminated and the result is

$$
\begin{aligned}
g_1 &= \{-1/1!\}(m_1), \\
g_2 &= \{1/2!\}(m_1^2) + \{-1/0!\}(m_2), \\
g_3 &= \{-1/3!\}(m_1^3) + \{1/1!\}(m_1m_2) + \{-1/0!\}(m_3), \\
g_4 &= \{1/4!\}(m_1^4) + \{-1/2!\}(m_1^2m_2) \\
&\quad + \{1/1!\}(m_1m_3 + [1/2]m_2^2) + \{-1/0!\}(m_4), \\
&\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned} \tag{20}
$$

Except for different numerical coefficients in braces and the interchange of the roles of the m's and g's, (20) is identical with (15). The discussion following (15) thus applies for finding the literal groups of (20). Table 1 is immediately adaptable with appropriate changes of notation. For any $\{t_{hk}\}$ of Table 1 except $\{t_{kk}\}$, the numerical coefficients for the inverse summation are $(-1)^h/[(k+h)-(k+1)]!$. For $\{t_{kk}\}$, the numerical coefficients are $(-1)^k/k!$.

6. Logarithms and Exponentials of Power Series. Expression 6.364 of Adams [1] is essentially

$$\text{(21)} \quad \ln(1 + g_1z + g_2z^2 + g_3z^3 + \cdots) = m_1z + m_2z^2 + m_3z^3 + \cdots .$$

Through a comparison of Adams' table accompanying 6.364 with (8) and (12) and an application of the reasoning of the last section, the arrangement of the m's and g's of (21) becomes that of Table 1. For any $\{t_{hk}\}$ of Table 1 except $\{t_{kk}\}$, the numerical coefficients are $(-1)^{h+1}$. For $\{t_{kk}\}$, the numerical coefficients are $(-1)^{k+1}/k$.

The exponential form is given as

$$\text{(22)} \quad \exp(g_1z + g_2z^2 + g_3z^3 + \cdots) = 1 + m_1z + m_2z^2 + m_3z^3 + \cdots .$$

Again through use of the methods of the last section, the arrangement of the m's and g's becomes that of Table 1. For any $\{t_{hk}\}$ of Table 1 except $\{t_{kk}\}$, the numerical coefficients for the exponential form are $1/[(k+h)-(k+1)]!$. For $\{t_{kk}\}$, the numerical coefficients are $1/k!$.

7. Calculation of Numerical Coefficients within a Literal Group. It becomes apparent that in any literal group except those which are coefficients of the $\{t_{kk}\}$'s the term $g_1^{h-1}g_{k-h+1}$ always appears. This term has weight k, degree h, and multiplier unity. The letter combinations in any literal group consist of all the combinations of g's which simultaneously have weight k and degree h. The quotient formed by $(h-1)!$ divided by the products of the factorials of the exponents of a combination of g's determines the bracketed numerical coefficient for that combination of g's. The single term of the g_1^k group (i.e., the coefficient of $\{t_{kk}\}$) has a multiplier of unity. This method for finding the terms of a literal group is essentially that outlined in Fielder [3].

8. Conclusions. It can be concluded that there are algebraic operations on polynomials or power series which have common forms for the literal groups. The braced coefficients are peculiar to particular algebraic operations. For a given operation, all numerical coefficients except $\{t_{kk}\}$ follow one combinatorial rule, while the $\{t_{kk}\}$ coefficients follow a different combinatorial rule. Since available tables of many algebraic operations which follow patterns similar to those discussed herein are terminated at index four or five, the information contained herein should be useful in extending the available tables. It is anticipated that

many other algebraic operations on polynomials can be found which are members of the operation class of this paper.

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Explicit Solutions of the One-Dimensional Heat Equation for a Composite Wall

By Marcia Ascher

1. Introduction. Explicit numerical solutions of the equation of heat conduction in a wall of one material have been widely discussed in the literature. Consideration of the forward difference equation studied in references [2], [3], [4], and [6] suggests a variety of ways to handle the solution for a composite wall. This paper is a study of the convergence, stability, comparative accuracy and comparative computing time of three explicit numerical solutions of the heat equation for a wall composed of two materials.

2. System of Equations. The equation for the one-dimensional flow of heat is:

$$c_s \rho_s \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_s \frac{\partial u}{\partial x}\right) \qquad \text{where } a_{s-1} \leqq x \leqq a_s \qquad s = 1, 2 \tag{1}$$

$$0 < t \leqq t_F$$

with the condition at the interface

$$k_1 \left(\frac{\partial u}{\partial x}\right)_1 = k_2 \left(\frac{\partial u}{\partial x}\right)_2 \tag{2}$$

where ρ_s, c_s, and k_s are constant with respect to time and temperature but may be different for each material.

We will assume the boundary conditions:

$$\begin{aligned} u(a_0, t) &= \text{constant}_1 \quad t \geqq 0 \\ u(a_2, t) &= \text{constant}_2 \quad t \geqq 0 \end{aligned} \tag{3}$$

and initial conditions:

$$u(x, 0) = \text{constant}_3 \quad a_0 < x < a_2 \tag{4}$$

Let each material's thickness, $a_s - a_{s-1}$, be divided into N_s equal parts of Δx_s, and t_F into equal parts of Δt_s. Let i denote the subscript associated with the space variable and j the subscript associated with the time variable. Let the solution of (1)–(4) be called $T(x, t)$.

Taylor series expansions of $T_{i,j+1}$, $T_{i+1,j}$, and $T_{i-1,j}$, about T_{ij} are used to obtain

$$T_{i,j+1} = \frac{r_s \Delta t}{\Delta x_s^2}\left[T_{i+1,j} - 2T_{ij} + T_{i-1,j}\right] + T_{ij} + E_1 \tag{5}$$

where $r_s = k_s/\rho_s c_s$ and

$$E_1 = \frac{\Delta t^2}{2}\frac{\partial^2 T}{\partial t^2} - \frac{r_s \Delta t \Delta x_s^2}{12}\frac{\partial^4 T}{\partial x^4} + \text{terms of higher order.}$$

Omitting E_1, equation (5) gives a difference equation for finding the approximate

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solution of (1)–(4), $T_{i,j+1}$, when x_i, x_{i-1}, and x_{i+1} are in the same material. This is the same as the forward difference equation for a one-material wall.

The following equation for $T_{i,j+1}$ at the interface is derived in a manner similar to that used by M. Lotkin [5] in his discussion of an implicit method for a wall of two materials.

$$(6)\qquad T_{ij} + \frac{\left[(T_{i+1,j} - T_{ij})\,\dfrac{k_2}{\Delta x_2} + (T_{i-1,j} - T_{ij})\,\dfrac{k_1}{\Delta x_1}\right]2\Delta t}{\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1} = T_{i,j+1} + E_2$$

where

$$E_2 = \frac{2\Delta t}{\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1} - \left[\frac{\Delta x_1 c_1 \rho_1 \Delta t}{4}\frac{\partial^2 T}{\partial t^2} - \frac{\Delta x_1^2 k_1}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_1 + \frac{k_1 \Delta x_1^3}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)_1 - \frac{\Delta x_2 c_2 \rho_2 \Delta t}{4}\frac{\partial^2 T}{\partial t^2} + \frac{\Delta x_2^2 k_2}{6}\left(\frac{\partial^3 T}{\partial x^3}\right)_2 + \frac{k_2 \Delta x_2^3}{24}\left(\frac{\partial^4 T}{\partial x^4}\right)_2\right] + \text{terms of higher order.}$$

Omitting E_2, equation (6) gives a difference equation for finding $T_{i,j+1}$ when x_i is at the interface, x_{i-1} is in the first material, and x_{i+1} is in the second material.

3. Definition of Methods. Stability is maintained in the explicit numerical solution for a wall of one material by choosing

$$(7)\qquad \Delta t \leqq \frac{\rho c \Delta x^2}{2k}.$$

Using equations (5) and (6) as our basic computing equations, three different means of choosing Δt will be defined and thereby different computational schemes. We will arbitrarily assume that $r_1 > r_2$ and, for simplicity, will confine the discussion to cases in which r_1/r_2 is an integer R, and $\sqrt{r_1/r_2}$ is an integer.

If Δx is specified as the thickness of each lamina within the wall, equation (7) gives two different maximum usable time increments depending on the properties of each material;

$$\Delta t_1 = \frac{\Delta x^2}{2r_1} \text{ and } \Delta t_2 = \frac{\Delta x^2}{2r_2}.$$

METHOD 1. In the first method an attempt is made to circumvent the difficulty of having two Δt's by letting $\Delta x_1 = \Delta x$ and redefining Δx_2 such that $\Delta x_2 = \Delta x_1/\sqrt{R}$. This increases the number of laminae in the second material but yields only one time increment, $\Delta t_1 = \Delta t_2 = \Delta t$. The computations would take place as follows:

Given $\Delta x = \Delta x_1$, r_1 and r_2

1. Compute $\Delta x_2 = \Delta x_1/\sqrt{R}$
2. Find $\Delta t = \Delta t_1 = \Delta t_2$
3. Set time equal to Δt
4. Use equation (5) to find the temperatures in material 1.
5. Use equation (6) to find the interface temperature.
6. Use equation (5) to find the temperatures in material 2.

7. Advance the time by Δt.
8. Repeat steps 4 to 7 until the temperatures at t_F have been computed.

METHOD 2. In the second method Δt is chosen by evaluating Δt_1 and Δt_2 and using whichever is less. The computations would then be:

Given $\Delta x = \Delta x_1 = \Delta x_2$, r_1 and r_2

1. Choose $\Delta t = \min(\Delta t_1, \Delta t_2)$
2. Proceed as in steps 3 to 8 of Method 1.

METHOD 3. In the third method both time increments are utilized by using the smaller increment only for those points at which it is necessary and the larger increment for the rest. The smaller time increment must be used for all points in the first material, at the interface, and for enough points in the second material to enable a smooth transition. For example, when $R = 4$, those points on the grid in Figure 1 denoted by dots are computed using Δt_1 and then those denoted by crosses are computed using Δt_2. The computations would proceed as follows:

Given $\Delta x = \Delta x_1 = \Delta x_2$, r_1 and r_2

1. Compute Δt_1
2. Compute Δt_2
3. Set time $= 0$
4. Set $Q = R - 1$
5. Advance time by Δt_1
6. Compute the temperatures in material 1 using equation (5) and Δt_1.
7. Compute the interface temperature using equation (6) and Δt_1.
8. If $Q = 0$ proceed to step 12.
9. Compute Q points in material 2 using equation (5) and Δt_1.
10. $Q - 1 \rightarrow Q$.
11. Repeat steps 5 to 10 until indicated by step 8.
12. Compute temperatures in material 2 using equation (5) and Δt_2.
13. Repeat steps 4 to 12 until the temperatures at t_F have been computed.

4. Convergence of Solutions.

THEOREM. *If there exists a solution of the system of equations* (1) *to* (4) *which has bounded derivatives* $\partial^2 T/\partial t^2$, $\partial^3 T/\partial x^3$, *and* $\partial^4 T/\partial x^4$ *in* $0 \leq t \leq t_F$, $a_0 \leq x < a_1$ *and*

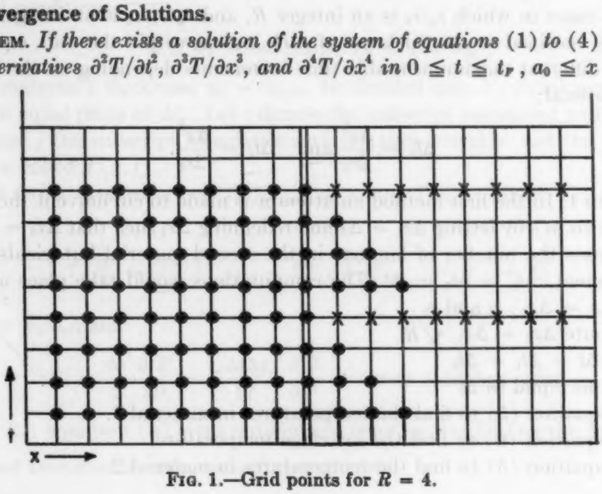

FIG. 1.—Grid points for $R = 4$.

$a_1 < x \leqq a_2$, then the solutions obtained with methods 1, 2, and 3 converge to the true solution. The rate of convergence is $O(\Delta x^2)$.

Proof. Let $B_1 = |$ upper bound on $\partial^2 T/\partial t^2 |$, $B_2 = |$ upper bound on $\partial^4 T/\partial x^4 |$, and $B_3 = |$ upper bound on $\partial^3 T/\partial x^3 |$. A barred derivative denotes that it is evaluated somewhere within the interval $0 \leqq t \leqq t_r$, $a_0 \leqq x < a_1$, $a_1 < x \leqq a_2$. Define the error at the point x_i, t_j to be $e_{ij} = T_{ij} - u_{ij}$. Here u_{ij} is the true solution of (1)–(4). The error arising from the use of equation (5) satisfies the following equation:

$$(8) \quad e_{i,j+1} = \frac{r_s \Delta t}{\Delta x_s^2}[e_{i+1,j} - 2e_{ij} + e_{i-1,j}] + e_{ij} + \Delta t^2 \overline{\frac{\partial^2 T}{\partial t^2}} - \frac{r_s \Delta t \Delta x_s^2}{12}\overline{\frac{\partial^4 T}{\partial x^4}}$$

$$(8') \quad e_{i,j+1} \leqq \left|\frac{r_s \Delta t}{\Delta x_s^2}\right| e_{i+1,j} + \left|1 - \frac{2r_s \Delta t}{\Delta x_s^2}\right| e_{ij} + \left|\frac{r_s \Delta t}{\Delta x_s^2}\right| e_{i-1,j} + \frac{\Delta t^2}{2} B_1 + \frac{r_s \Delta t \Delta x_s^2}{12} B_2.$$

Let $\alpha_j = \max_i | e_{ij} |$, then

$$(8'') \quad e_{i,j+1} \leqq \left[\left|\frac{r_s \Delta t}{\Delta x_s^2}\right| + \left|1 - \frac{2r_s \Delta t}{\Delta x_s^2}\right| + \left|\frac{r_s \Delta t}{\Delta x_s^2}\right|\right] \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{r_s \Delta t \Delta x_s^2}{12} B_2.$$

In methods 1 and 2, $\Delta t \leqq \Delta x_s^2/2r_s$ $(s = 1, 2)$ and therefore $r_s \Delta t/\Delta x_s^2 \leqq \frac{1}{2}$. This causes each of the terms within the absolute value signs in equation (8″) to be positive and so they may be eliminated giving

$$(9) \quad e_{i,j+1} \leqq \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{\Delta x_s^4}{24} B_2.$$

In method 3, the larger time increment, Δt_2, is only used at points for which $\Delta t_2 \leqq \Delta x_2^2/2r_2$ is satisfied and whenever Δt_1 is used, in the first material or for the transition values, $\Delta t_1 \leqq \Delta x_s^2/2r_s$ $(s = 1, 2)$. Therefore, whenever equation (5) is used $\Delta t \leqq \Delta x^2/2r$ and so equation (9) also applies to method 3.

The error in the evaluation of the interface temperature from equation (6) satisfies

$$(10) \quad e_{i,j+1} = e_{i,j} + \frac{\left[(e_{i+1,j} - e_{ij})\dfrac{k_2}{\Delta x_2} + (e_{i-1,j} - e_{ij})\dfrac{k_1}{\Delta x_1}\right] 2\Delta t}{\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1} - E_2$$

$$(11) \quad \begin{aligned} e_{i,j+1} \leqq\ & \alpha_j \left[\frac{k_2}{\Delta x_2} + \frac{k_1}{\Delta x_1}\right]\left[\frac{2\Delta t}{\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1}\right] \\ & + \left| 1 - \frac{2\Delta t\left[\dfrac{k_2}{\Delta x_2} + \dfrac{k_1}{\Delta x_1}\right]}{\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1}\right| \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{\Delta t B_3(\Delta x_1^2 k_1 + \Delta x_2^2 k_2)}{3(\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1)} \\ & + \frac{\Delta t B_2(k_1 \Delta x_1^3 + k_2 \Delta_2^3)}{12(\Delta x_2 c_2 \rho_2 + \Delta x_1 c_1 \rho_1)}. \end{aligned}$$

Since in all three methods the Δt used in the interface equation is min $(\Delta t_1, \Delta t_2)$,

$$\frac{\Delta x_2 c_2 \rho_2}{\Delta t} + \frac{\Delta x_1 c_1 \rho_1}{\Delta t} \geqq 2\left[\frac{k_2}{\Delta x_2} + \frac{k_1}{\Delta x_1}\right]$$

and, therefore, (11) can be rewritten as

$$e_{i,j+1} \leqq \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{B_3}{6}\left[\frac{k_1 \Delta x_2(\Delta x_1^3) + k_2 \Delta x_1(\Delta x_2^3)}{\Delta x_1 k_2 + \Delta x_2 k_1}\right] \tag{12}$$
$$+ \frac{B_2}{24}\left[\frac{k_1 \Delta x_2(\Delta x_1^4) + k_2 \Delta x_1(\Delta x_2^4)}{\Delta x_1 k_2 + \Delta x_2 k_1}\right].$$

Let $\Delta x = \max(\Delta x_1, \Delta x_2)$, then (12) becomes

$$e_{i,j+1} \leqq \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{\Delta x^3}{6} B_3 + \frac{\Delta x^4}{24} B_2. \tag{13}$$

Comparing equations (9) and (13), it can be seen that

$$\alpha_{j+1} \leqq \alpha_j + \frac{\Delta t^2}{2} B_1 + \frac{\Delta x^3}{6} B_3 + \frac{\Delta x^4}{24} B_2 \quad \text{where} \quad \Delta t = \max(\Delta t_1, \Delta t_2) \tag{14}$$

and $\Delta x = \max(\Delta x_1, \Delta x_2)$. At any point $t = j\Delta t$

$$\alpha_j \leqq \alpha_0 + j\left[\frac{\Delta t^2}{2} B_1 + \frac{\Delta x^3}{6} B_3 + \frac{\Delta x^4}{24} B_2\right]. \tag{15}$$

The rate of convergence is, therefore, of the order $0(\Delta t \ \& \ \Delta x^2)$. Since Δt is of the order $0(\Delta x^2)$, the rate of convergence is $0(\Delta x^2)$.

5. Analytical Example. In order to examine the performance of the three methods, a test case will be used for which some analytical solutions are known. The equations for the composite wall will first be reduced, by transformations of the variables, to the equations for a wall of one material. To do this we will impose the conditions:

$$\begin{aligned} &\text{(i)} && k_1 c_1 \rho_1 = k_2 c_2 \rho_2 \\ &\text{(ii)} && a_0 = 0. \end{aligned} \tag{16}$$

Define the transformations

$$\begin{aligned} &\text{(i)} && y = \begin{cases} \left[b + (1-b)\dfrac{a_2}{a_1}\right] x & 0 \leqq x \leqq a_1 \\ bx + (1-b)a_2 & a_1 \leqq x \leqq a_2 \end{cases} \\ &\text{(ii)} && \tau = r_2 b^2 t \qquad 0 \leqq t \leqq t_r \end{aligned} \tag{17}$$

$$\text{where} \quad b = \frac{k_1 a_2}{k_1(a_2 - a_1) + k_2 a_1}.$$

This reduces equations (1) to (4) to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2} \qquad 0 \leqq y \leqq a_2, 0 \leqq \tau \leqq t_r r_2 b^2 \tag{18}$$

$$\begin{aligned} u(0, \tau) &= \text{constant}_1 \quad \tau \geqq 0 \\ u(a_2, \tau) &= \text{constant}_2 \quad \tau \geqq 0 \end{aligned} \tag{19}$$

$$u(y, 0) = \text{constant}_3 \quad 0 < y < a_2. \tag{20}$$

The solution to this set of equations is found in reference [6] for $\tau = 0(.005).1$, $y = .4$, $\text{constant}_1 = \text{constant}_2 = 0$, $\text{constant}_3 = 1$, $a_2 = 1$.

6. Test Case Results. Each method was programmed for the IBM 704 EDPM. The test data used was $k_1 = 10$, $c_1 = 5$, $\rho_1 = 2$, $k_2 = 5$, $c_2 = 4$, $\rho_2 = 5$, $a_0 = 0$, $a_1 = .5$, and $a_2 = 1$. Each method was run for three cases: Case A, $\Delta x = .1$; case B, $\Delta x = .05$, and case C, $\Delta x = .025$.

To examine the rate of convergence, the maximum differences for a given time point were found between the results for case A and case C, and the results for case B and case C. The ratios of these maximum differences ranged between 3.8 and 5.0 for each of the three methods. Since the ratios of the Δx's were 2.0, this would seem to corroborate that the rate of convergence is $0(\Delta x^2)$.

To compare the accuracy of the three methods, the data presented in reference [6] was used. Their values correspond, according to the transformation presented in equation (17), to $x = .55$ and $x = .7$ for $t = 0\ (.01125).225$. These values, as well as those obtained for case C for the three methods, are presented in Table 1. It can be seen from the table that, although they all showed close agreement, method 2 gave the most accuracy with a maximum of .08% error, method 1 the next with a maximum of .14% error, and method 3 the least with a maximum of .28% error. The symmetry of the transformed equation indicates that for this case the temperatures should be the same for $x = .15$ and $x = .925$, $x = .3$ and $x = .85$, $x = .45$ and $x = .775$, and for $x = .55$ and $x = .7$. When comparing the results at these points at $t = .1$, method 1 has them all the same, method 2 has a greatest difference of .04%, and method 3 has a greatest difference of .53%. A comparison at $t = .225$ shows method 1 has them all the same, method 2 still with a greatest difference of .04%, and method 3 with a greatest difference of .20%. These differences are reasonable in terms of the methods of choosing Δt for computation and illustrate that in method 1 the results for both materials is equally accurate, for method 2 the results for the second material is a bit more accurate than for the first material, while for method 3 the result for the second material is less accurate than for the first but their differences decrease as more time steps are taken.

TABLE 1

Time	*Exact Solution*	*Method 1, $x = .55$*	*Method 1, $x = .7$*	*Method 2, $x = .55$*	*Method 2, $x = .7$*	*Method 3, $x = .55$*	*Method 3, $x = .7$*
.0225	.9953	.99536	.99536	.99551	.99499	.99579	.99557
.0450	.9518	.95145	.95145	.95188	.95127	.95254	.95055
.0675	.8832	.88261	.88261	.88306	.88275	.88354	.88136
.0900	.8088	.80807	.80807	.80851	.80840	.80868	.80686
.1125	.7363	.73556	.73556	.73598	.73597	.73590	.73448
.1350	.6686	.66782	.66782	.66821	.66826	.66797	.66689
.1575	.6063	.60561	.60561	.60597	.60605	.60562	.60481
.1800	.5496	.54891	.54891	.54924	.54933	.54882	.54821
.2025	.4981	.49739	.49739	.49770	.49780	.49724	.49678
.2250	.4513	.45067	.45067	.45095	.45105	.45047	.45011

Although the time increments were chosen on the basis of equation (7), it is pointed out in reference [6] that a slightly larger increment is possible, namely

$$\Delta t_{\max} \leqq \frac{c\rho\, \Delta x^2}{2k \sin^2\left[\dfrac{(N-1)\pi}{2N}\right]}. \tag{21}$$

Using case A, the programs were run until instability appeared in an attempt to see what the maximum increment actually was for two materials. For method 3, if the maximum Δt's are computed separately for each material of 5 laminae with equation (21), the result is $\Delta t_1 \leqq .005528$ and $\Delta t_2 \leqq .022112$. The experimental results corroborated this since it was stable up to $\Delta t_1 = .0055$ and $\Delta t_2 = .0220$ but unstable for $\Delta t_1 = .0056$ and $\Delta t_2 = .0230$. For method 2, computing the maximum time increment for each material of 5 laminae and then choosing the smaller, one gets $\Delta t \leqq .005528$. The test showed the same result as it was stable up to $\Delta t = .0055$ but unstable for $\Delta t = .0056$. In method 1, when using equation (7), the adjustment of the thicknesses of the laminae in the second material led to equal Δt's. However, when using equation (21), the Δt's obtained are $\Delta t_1 \leqq .005528$ and $\Delta t_2 \leqq .005125$ since the number of laminae in each material is different. The test runs showed that it remained stable until $\Delta t = .0053$ but was unstable with $\Delta t = .0054$. This might indicate that a maximum time increment was used which is the average of Δt_1 and Δt_2 but no conclusion is possible since the stability condition states that it should be stable below the computed $\Delta t_{\max}$ but it is not necessarily unstable for a Δt above it. However, all these stability test runs seem to indicate that when there are two materials in a wall, the maximum usable time increment is quite closely related to the maximum increments computed for each material separately.

7. Comparison of Computing Time Required. The amount of computing time required for each method can be compared by comparing the number of temperatures that must be evaluated.

Let us assume that the thicknesses of the first and second materials are equal and that $t_r = PR\Delta t_1$ (where P is any integer). It should be noted that given Δx, r_1, and r_2, each method will compute the same value for Δt_1 and Δx_1.

In method 1 there are $N_1(1 + \sqrt{R})$ laminae and PR time points. Therefore, the number of temperatures computed equals $P[RN_1(1 + \sqrt{R}) - R]$. In method 2 there are $2N_1$ laminae and PR time points and so $P[2N_1R - R]$ temperatures are computed. In method 3 each of the $N_1 - 1$ temperatures in the first material

TABLE 2

	Case A	Case B	Case C
Method 1	P (56)	P (116)	P (236)
Method 2	P (36)	P (76)	P (156)
Method 3	P (30)	P (55)	P (105)
Ratio M1:M2:M3	1.87:1.20:1.00	2.11:1.38:1.00	2.25:1.49:1.00

is computed PR times, the interface is computed PR times, and for each step of Δt_2, which occurs P times, $N_1 - 1$ values are computed plus the additional number of transition values $= \sum_{i=1}^{R-1} i$. Method 3, therefore, computes

$$P\left[N_1 R + N_1 - 1 + \frac{R^2}{2} - \frac{R}{2}\right]$$

values. From this it can be seen that for all $R > 1$ method 2 is faster than method 1. For $2N_1 - 2 > R > 1$ method 3 is faster than method 2. This comparison is illustrated in Table 2 for cases A, B and C of the test data used.

8. Conclusions. The analysis and test cases used considered constant boundary and initial conditions. Since the stability and convergence depend also on the boundary and initial conditions, as has been pointed out in references [1], [3], and [4], it is quite possible that the introduction of varying conditions would lead to different results as to the usefulness of each method.

From this study it seems that method 1 is the least acceptable since it takes the most computing time, gives less accuracy than method 2, and presents the most difficulty when R is not an integer. Depending on the amount of accuracy desired, methods 2 and 3 seem of equal usefulness since method 2 gives the most accuracy but method 3 takes less computing time.

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A Method of "Alternating Corrections" for the Numerical Solution of Two-Point Boundary Value Problems

By David A. Pope

Abstract. In this paper a method of "alternating corrections" is defined and analyzed for the numerical solution of the two-point boundary value problem

$$\begin{aligned} y'' &= f(x, y) \\ y(0) &= a \\ y(1) &= b. \end{aligned} \tag{0.1}$$

The case where the first derivative does not enter explicitly into the differential equation is chosen for simplicity of treatment. The alternating corrections method can easily be modified to treat the more general case. The function $f(x, y)$ is assumed to have continuous second derivatives, but the differential equation may, of course, be non-linear.

The method to be described is essentially a relaxation technique suitable for an automatic digital computer. The main feature of the method is that most of the "correcting" is done in the early stages of the computation, using a small number of points; thus a rough approximation to the solution is obtained quickly. This approximation can then be made more accurate in the later stages of the computation, as the number of points is increased.

In Section 1 the method is described. Section 2 gives a rigorous truncation and stability analysis. Section 3 contains the proof of the convergence of the method giving an estimate of the rate of convergence, and in Section 4 some experimental results obtained on a digital computer are examined.

1. Definition of the Method. In the following, we will denote by R a closed and bounded region of the x-y plane, in which we will assume both the solution to (0.1) and the approximations to that solution are known to lie, a priori. (See Collatz [1] p. 188 for a sufficient condition for the existence of the solution to this problem.) The function $f(x, y)$ is assumed to be continuous and to have continuous first and second derivatives in R. The method to be discussed consists of two stages, as follows:

A. *Interpolation by Halves.* Suppose the interval [0, 1] is partitioned into n equal parts by $n + 1$ equally spaced points, and an approximation y_j to the solution of (0.1) is defined at these points. We then refine the partition by subdividing [0, 1] into $2n$ equal parts, by $2n + 1$ equally spaced points x_j, $j = 0, 1, 2, \cdots 2n$. Then the points of the original partition are given by the x_j with even index. We then interpolate y_j for the odd indices by using the explicit formula

$$y_j = \frac{1}{2}[y_{j+1} + y_{j-1}] - \frac{h^2}{4}[f(x_{j+1}, y_{j+1}) + f(x_{j-1}, y_{j-1})], \tag{1.1}$$

$$j = 1, 3, 5, \cdots 2n - 1.$$

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We shall abbreviate the right-hand side of (1.1) by the operator $K(y_j)$. Here, and in the following, $h = \Delta x_j = 1/2n$.

B. *Alternating Corrections.* After step A, the values of y_j with odd and even index are corrected alternately by using the same formula (1.1). Thus we have

$$y_j^{s+1} = K(y_j^s) \qquad \text{for } j \text{ odd,} \tag{1.2}$$

$$y_j^{s+1} = K(y_j^{s+1}) \qquad \text{for } j \text{ even, } j \neq 0,\ j \neq 2n, \tag{1.3}$$

$$y_0^{s+1} = a, \tag{1.4}$$

$$y_{2n}^{s+1} = b, \tag{1.5}$$

where y_j^s is the value of y_j at the s-th iteration of step B. Then, as we shall show in Section 3, as $s \to \infty$, the values y_j^s approach the solution of the system of difference equations

$$\begin{aligned} y_j &= K(y_j) \qquad j = 1, 2, \cdots, 2n-1, \\ y_0 &= a \\ y_{2n} &= b. \end{aligned} \tag{1.6}$$

To start the computation, we usually will set $n = 1$, and $y_0 = a$, $y_1 = b$. Then step A interpolates a value at $x = \frac{1}{2}$, and we renumber the values y_0, y_1, y_2. Here step B is not needed, so we perform step A again, getting now five values. At this point we perform step B a number of times, until sufficient convergence to (1.6) for our purpose is obtained. We continue in this way, doubling the number of points with step A, then following this with a number of iterations of step B, until we have the desired accuracy. In Section 3 we will consider some estimate of the number of iterations of step B necessary for a given accuracy.

2. Stability and Truncation Error. In this section we shall give a rigorous estimate of the truncation error in and stability of the difference equations (1.6). We note that the global truncation error is of order h^{-2} times the local truncation error, rather than h^{-1} times, as might be expected from a naive analysis.

In the following, we let $Y(x)$ be the exact solution to the differential equation (0.1). Let $Y_j = Y(x_j)$, and let y_j be the exact solution to the system of difference equations (1.6). Let the error $e_j = Y_j - y_j$. Then, using the law of the mean, we obtain from (1.6) the system of difference equations

$$\begin{aligned} e_j = \frac{1}{2}[e_{j+1} + e_{j-1}] - \frac{h^2}{4}[e_{j+1} fy(x_{j+1}, \eta_{j+1}) \\ + e_{j-1} fy(x_{j-1}, \eta_{j-1})] + t_j, \qquad j = 1, 2, \cdots 2n-1, \end{aligned} \tag{2.1}$$

and

$$e_0 = e_{2n} = 0, \tag{2.2}$$

where η_j is between Y_j and y_j. Here the local truncation error

$$t_j = -\frac{5h^4}{24} y^{iv}(\xi_j), \qquad x_{j-1} < \xi_j < x_{j+1},$$

which is obtained from Taylor's formula. Rewriting (2.1) in matrix form, we have

$$Ae = -\frac{h^2}{4} Fe + t \tag{2.3}$$

where e is the column error vector whose transpose is $(e_1, e_2, \cdots e_{2n-1})$, t the truncation error vector with transpose $(t_1, t_2, \cdots t_{2n-1})$, A is the second difference matrix

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \cdots \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots \\ \cdots & & & & \end{bmatrix}$$

and F is the matrix

$$F = \begin{bmatrix} 0 & g_{12} & 0 & 0 & \cdots \\ g_{21} & 0 & g_{23} & 0 & \cdots \\ 0 & g_{32} & 0 & g_{33} & \cdots \\ \cdots & & & & \end{bmatrix}$$

where $g_{ij} = f_y(x_i, \eta_j)$, evaluated at the intermediate points given in (2.1). It is well known that the matrix A has eigenvectors v_j with components $\sin \frac{jm\pi}{2n}$, $m = 1, 2, \cdots 2n - 1$, and corresponding eigenvalues $\lambda_j = 1 - \cos \frac{j\pi}{2n}$ $j = 1, 2, \cdots 2n - 1$. Hence A^{-1} exists, and its largest eigenvalue is $\left(1 - \cos \frac{\pi}{2n}\right)^{-1}$. Therefore, multiplying (2.3) by A^{-1}, we have

$$\left(I + \frac{h^2}{4} A^{-1}F\right) e = A^{-1}t. \tag{2.4}$$

Now we can prove two lemmas giving estimates of the error.

LEMMA 1. *Suppose $f_y > 0$ in the region R. Then the components of the error vector e satisfy the inequality*

$$| e_i | \leqq \frac{5h^2 \max | y^{iv} |}{12 \min f_y} \qquad i = 1, 2, \cdots 2n - 1, \tag{2.5}$$

where the extreme values are taken over the region R.

Proof. Let the norm $\| e \| = \max_i | e_i |$, and the subordinate matrix norm $\| A \| = \max_i \sum_j | a_{ij} |$. (Cf. Faddeeva [2] p. 58.) Rewriting (2.3) in the form

$$\left(A + \frac{h^2}{4} F\right) e = (I - B) e = t, \tag{2.6}$$

we have defined the matrix $B = I - A - \frac{h^2}{4} F$. But since $f_y > 0$, we can conclude

that $\| B \| \leqq 1 - \frac{h^2}{2} \operatorname{Min} f_y < 1$, where the minimum is taken over the closed region R. Hence the series $I + B + B^2 + \cdots$ converges to $(I - B)^{-1}$, and $\| (I - B)^{-1} \| \leqq (1 - \| B \|)^{-1}$. Also we note that $\| t \| \leqq \frac{5h^4}{24} \max | y^{iv} |$.

Using these estimates and (2.6) gives us

$$\| e \| \leqq \frac{\| t \|}{1 - \|B\|} \leqq \frac{5h^4}{24} \max | y^{iv} | \cdot \frac{2}{h^2 \min f_y}$$

from which inequality (2.5) follows.

The second lemma takes care of the case when f_y is negative or zero in R. Here we estimate the root-mean- square of the error.

LEMMA 2. *Suppose* $| f_y | < \pi^2$ *in the region* R. *Then, for* h *sufficiently small, the error vector satisfies*

$$\frac{\| e \|}{\sqrt{2n - 1}} \leqq \frac{5h^2 \max | y^{iv} |}{12\pi^2(1 - \pi^{-2}\max | f_y |)} + O(h^4). \tag{2.7}$$

Proof. In this proof, we use the euclidean norm, and the subordinate matrix norm $\|A\|$ equal to the square root of the largest eigenvalue of AA^T (Faddeeva [2] p. 59). We see by inspection that FF^T has a maximum row sum not exceeding $\max 4f_y^2$. But we know that its largest eigenvalue does not exceed this maximum row sum. This gives the estimate

$$\|F\| < 2 \max | f_y | . \tag{2.8}$$

Also, from the eigenvalues of the symmetric matrix A we obtain

$$\|A^{-1}\| = (1 - \cos \pi h)^{-1}. \tag{2.9}$$

Using this together with (2.8) we get

$$\left\| \frac{h^2}{4} A^{-1}F \right\| \leqq \frac{h^2 \max | f_y |}{2(1 - \cos \pi h)} . \tag{2.10}$$

But for small h, we have the estimate

$$(1 - \cos \pi h)^{-1} = \frac{2}{\pi^2 h^2} - \frac{1}{6} + O(h^2), \tag{2.11}$$

hence, for h sufficiently small,

$$\left\| \frac{h^2}{4} A^{-1}F \right\| \leqq (\pi^{-2} + O(h^2)) \max | f_y | < 1, \tag{2.12}$$

since $\max | f_y | < \pi^2$. Also, with this norm, $\|t\| \leqq \frac{5}{24} \operatorname{Max} | y^{iv} | \sqrt{2n + 1}$.

Therefore, by the same reasoning as in Lemma 1, $\left(I + \frac{h^2}{4} A^{-1}F\right)^{-1}$ exists, and its norm does not exceed $\left(1 - \left\| \frac{h^2}{4} A^{-1}F \right\|\right)^{-1}$. Putting these estimates into (2.4), we get (2.7), proving the lemma.

It should be noted that the restriction $-f_y < \pi^2$ is a natural one, as can be seen from an examination of the boundary value problem

$$\begin{aligned} y'' &= -Ky \\ y(0) &= 0 \\ y(1) &= 1. \end{aligned} \tag{2.13}$$

Here, of course, the solution does not exist when $K = -f_y = \pi^2$, as we have an eigenvalue problem.

3. Convergence of the Alternating Corrections Method. In this section we shall give a proof of the convergence of the alternating corrections method (method B of Section 1). Following the notation of Section 1, we shall denote the value of the approximation at the point x_j for the s-th iteration by y_j^s, and the exact solution to the difference equations (1.6) by y_j. Then we define the error $\epsilon_j^s = y_j - y_j^s$. Then we have, for j odd,

$$\epsilon_j^{s+1} = \frac{1}{2}[\epsilon_{j+1}^s + \epsilon_{j-1}^s] - \frac{h^2}{4}[\epsilon_{j+1}^s f_y(x_{j+1}, \eta_{j+1}) + \epsilon_{j-1}^s f_y(x_{j-1}, \eta_{j-1})], \tag{3.1}$$

where η_j is between y_j and y_j^s, for each j. For j even, $j \neq 0$, $j \neq 2n$,

$$\epsilon_j^{s+1} = \frac{1}{2}[\epsilon_{j+1}^{s+1} + \epsilon_{j-1}^{s+1}] - \frac{h^2}{4}[\epsilon_{j+1}^{s+1} f_y(x_{j+1}, \xi_{j+1}) + \epsilon_{j-1}^{s+1} f_y(x_{j-1}, \xi_{j-1})], \tag{3.2}$$

and for the endpoints,

$$\epsilon_0^s = \epsilon_{2n}^s = 0. \tag{3.3}$$

Now we define $\mu = \max \left| 1 - \frac{h^2}{2} f_y \right|$, the maximum being taken over the region R. Then for j odd, we have

$$| \epsilon_j^{s+1} | \leqq \tfrac{1}{2}\mu[| \epsilon_{j+1}^s | + | \epsilon_{j-1}^s |] \tag{3.4}$$

and for j even, $j \neq 0$, $j \neq 2n$

$$| \epsilon_j^{s+1} | \leqq \tfrac{1}{2}\mu[| \epsilon_{j+1}^{s+1} | + | \epsilon_{j-1}^{s+1} |]. \tag{3.5}$$

To estimate the error ϵ_j^s, we majorize it with a quantity E_j^s, defined recursively as follows:

$$E_j^0 = | \epsilon_{2j}^0 | \qquad j = 0, 1, \cdots n. \tag{3.6}$$

$$E_j^{s+1} = \tfrac{1}{4}\mu^2[E_{j-1}^s + 2E_j^s + E_{j+1}^s] \qquad j = 1, 2, \cdots n - 1. \tag{3.7}$$

$$E_0^{s+1} = E_n^{s+1} = 0. \tag{3.8}$$

In this section we will use the euclidean norms $\| \epsilon \|^2 = \sum_0^{2n} \epsilon_j^2$ and

$$\| E \|^2 = \sum_0^m E_j^2.$$

Lemma 3 gives an estimate for $\| \epsilon \|$ in terms of $\| E \|$.

LEMMA 3. *For every $s \geqq 1$,*

$$\| \epsilon^s \|^2 \leqq (1 + \mu^2) \| E^s \|^2. \tag{3.9}$$

Proof. First the inequality $| \epsilon_{2j}^s | \leqq E_j^s$ is established by induction on s. For $s = 0$, the inequality holds by definition. Assuming the inequality is true for s, the proof for $s + 1$ follows from the inequalities

$$\begin{aligned} | \epsilon_{2j}^{s+1} | &\leqq \tfrac{1}{2}\mu[| \epsilon_{2j-1}^{s+1} | + | \epsilon_{2j+1}^{s+1} |] \\ &\leqq \tfrac{1}{4}\mu^2[| \epsilon_{2j-2}^s | + 2 | \epsilon_{2j}^s | + | \epsilon_{2j+2}^s |] \\ &\leqq E_j^{s+1}. \end{aligned}$$

Now for $2j + 1$ we have the inequality

$$\begin{aligned} | \epsilon_{2j+1}^s | &\leqq \tfrac{1}{2}\mu[| \epsilon_{2j}^s | + | \epsilon_{2j+2}^s |] \\ &\leqq \tfrac{1}{2}\mu[E_j^s + E_{j+1}^s]. \end{aligned}$$

Combining these, and using the triangle inequality, we obtain

$$\begin{aligned} \| \epsilon^s \|^2 &\leqq \sum_0^n | E_j^s |^2 + \frac{1}{4}\mu^2 \sum_0^{n-1} | E_j^s + E_{j+1}^s |^2 \\ &\leqq (1 + \mu^2) \| E^s \|^2, \end{aligned} \tag{3.10}$$

which proves Lemma 3.

We now expand E_j^s in a finite Fourier sine series, with Fourier coefficients given by

$$F_m^s = \frac{2}{n} \sum_{j=1}^{n-1} E_j^s \sin \frac{mj\pi}{n}, \qquad m = 1, 2, \cdots n - 1. \tag{3.11}$$

Then we have the expansion

$$E_j^s = \sum_{m=1}^{n-1} F_m^s \sin \frac{mj\pi}{n}, \qquad j = 0, 1, \cdots, n. \tag{3.12}$$

Substitution of (3.12) into (3.7) now yields the recursion relation

$$F_m^{s+1} = \frac{1}{2}\mu^2 \left(1 + \cos \frac{m\pi}{n}\right) F_m^s. \tag{3.13}$$

From (3.13) we then obtain the estimate

$$\| E^s \| \leqq \left[\frac{1}{2}\mu^2 \left(1 + \cos \frac{\pi}{n}\right)\right]^s \| E^0 \|, \tag{3.14}$$

and applying Lemma 3, we have the final estimate

$$\| \epsilon^s \| \leqq (1 + \mu^2)^{1/2} \left[\frac{1}{2}\mu^2 \left(1 + \cos \frac{\pi}{n}\right)\right]^s \| \epsilon^0 \|. \tag{3.15}$$

Therefore we have proved

LEMMA 4. *If $-1 < \rho = \frac{1}{2}\mu^2 \left(1 + \cos \frac{\pi}{n}\right) < 1$, the alternating corrections method*

will converge geometrically to the solution of the difference equations (1.6), *with convergence factor* ρ.

For small h, we note that

$$\rho = \max\,[1 - h^2(f_y + \pi^2) + O(h^4)] \tag{3.16}$$

and again we have the natural restriction for convergence mentioned at the end of Section 2.

The number of iterations needed at each stage in the method can now be estimated as follows. Clearly, the convergence factor increases as $h \to 0$, hence the convergence is much faster for large values of h. On the other hand, it would be futile to carry the iterations so far that $\| \epsilon \|$ is much smaller than $\| e \|$, as we are interested not in the solution of (1.6), but of (0.1). Hence a useful compromise might be to iterate the alternating corrections until $\| \epsilon \|$ and $\| e \|$ are approximately equal, then to interpolate, and start again with interval $h/2$. If this scheme is followed, we would want to cut the error ϵ by a factor of about $\frac{1}{4}$ by iteration after each interpolation, since the error $\| e \|$ is of order h^2. Then we have $\rho^s = \frac{1}{4}$, which gives us the approximation

$$s = -\frac{\log 4}{\log \rho} \approx \frac{\log 4}{h^2(f_y + \pi^2)}\,.$$

This shows that it would take about four times as many iterations for the next stage, after interpolation by halves. Since there are about twice as many points, the total amount of computational work is multiplied by eight at each succeeding stage. Clearly this process cannot be used for very many stages.

In practice h will probably not be made less than 2^{-8} or 2^{-9}, and if more accuracy is needed, a more sophisticated set of difference equations than (1.6) would be used. The alternating corrections method, however, is excellent for obtaining

TABLE 1

h	error at $x = \frac{1}{2}$	number of iterations
2^{-1}	.052083	0
2^{-2}	.013021	27
2^{-3}	.003255	100
2^{-4}	.000814	329
2^{-5}	.000203	1026
2^{-6}	.000051	2948

TABLE 2

h	error at $x = \frac{1}{2}$	number of iterations
2^{-1}	.065074	0
2^{-2}	.013935	23
2^{-3}	.003366	82
2^{-4}	.000834	269
2^{-5}	.000208	831
2^{-6}	.000052	2339

a good approximation quickly, which could be used as a first guess in a more complicated relaxation scheme.

4. Some Experimental Results. In this section we shall discuss the results of two problems which were computed using the alternating corrections method. The computation was done using the Univac Scientific 1103 computer at the University of Minnesota Scientific Computing Laboratory.

The first problem was the linear equation

$$y'' = 2x^2$$
$$y(0) = 0 \tag{4.1}$$
$$y(1) = 1.$$

In Table 1 the results of this computation are summarized.

Formula (2.7) with $f_y = 0$, $y^{iv} = 4$ gives the r.m.s. error $< .16887h^2$, in good agreement with the error at $x = \frac{1}{2}$.

For each value of h, stage B was iterated until there was no change larger than 2^{-29} in any y_j^s. The number of iterations necessary to accomplish this is also given in Table 1. If we use the estimate $\| \epsilon^0 \| = .16887h^2$, we get the relation

$$.16887h^2\rho^s = 2^{-29},$$

from which we get the approximation

$$s \approx h^{-2}(1.856 + .2026 \log h),$$

which agrees well with the number of iterations actually performed.

The second problem tried was the nonlinear equation

$$y'' = 2y^2$$
$$y(0) = 0 \tag{4.2}$$
$$y(1) = 1$$

Table 2 gives a summary of this computation.

The fact that the global truncation error is of order h^2 is again displayed in Table 2. The number of iterations necessary at each stage was governed by the same scheme as in problem 1, but with the criterion 2^{-31} instead of 2^{-29}.

Finally, it may be noted that the approximations generated by the alternating corrections method could be improved greatly by a "deferred approach to the limit", using the approximations obtained from the last two values of h computed.

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Note on the Asymptotic Expansion of the Modified Bessel Function of the Second Kind

By E. Dempsey and G. C. Benson

Tables of modified Bessel functions of the second kind, $K_n(z)$, for integral order n and for special forms of real positive argument z have recently been computed [1]. To obtain the desired accuracy throughout the tables it was found necessary to look into the methods available for approximating the remainder term R_p in the asymptotic expansion of the function:

$$K_n(z) = \sum_{m=0}^{p-1} u_m + (R_p/u_p)u_p \tag{1}$$

where

$$u_m = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \frac{e^{-z}}{m!(2z)^m} \tag{2}$$

u_{p-1} is usually the *least term* (*in absolute value*) of the series. Since the converging factor R_p/u_p may itself be developed as an asymptotic series, it is convenient to discuss this quantity rather than the remainder term.

In a treatment due to Dingle a general theory of asymptotic expansions and their converging factors [2] is applied to the hypergeometric function [3], special cases of which are the modified Bessel functions of the first and second kinds. The formula relevant to the latter case is as follows

$$\frac{R_p}{u_p} = \sum_{t=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+p+\frac{1}{2}-t)}{\Gamma(n+\frac{1}{2}-t)\Gamma(n+p+\frac{1}{2})} (-2z)^t \Lambda_{p-n-\frac{1}{2}}^{(t)}(2z) \tag{3}$$

in which u_p is now not necessarily the first term after the *least* term of the series in Eq (1) but may be a term some way beyond this. Going to the fifth term in Eq (3) the functions $\Lambda_{p-n-\frac{1}{2}}^{(t)}(2z)$ are defined by

$$\begin{aligned}\Lambda_s^{(0)}(s+\theta) &= \frac{1}{2} - \frac{1}{8s}(1-2\theta) + \frac{1}{32s^2}(1-2\theta-4\theta^2) \\ &+ \frac{1}{128s^3}(1+6\theta+8\theta^2+8\theta^3) - \frac{1}{512s^4}(13+22\theta+20\theta^2+16\theta^3+16\theta^4)\end{aligned} \tag{4}$$

$$\begin{aligned}\Lambda_s^{(1)}(s+\theta) &= \frac{1}{4s} - \frac{1}{16s^2}(1+4\theta) + \frac{1}{64s^3}(3+8\theta+12\theta^2) \\ &- \frac{1}{256s^4}(11+20\theta+24\theta^2+32\theta^3)\end{aligned} \tag{5*}$$

$$\Lambda_s^{(2)}(s+\theta) = -\frac{1}{8s^2} + \frac{1}{16s^3}(1+3\theta) - \frac{1}{128s^4}(5+12\theta+24\theta^2) \tag{6}$$

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* The coefficient of $(64s^3)^{-1}$ in $\Lambda_s^{(1)}$ given in ref. [3] is incorrect; the correct form is given above.

(7) $\Lambda_s^{(3)}(s+\theta) = \frac{1}{16s^3} - \frac{1}{32s^4}(1+4\theta)$

(8) $\Lambda_s^{(4)}(s+\theta) = -\frac{1}{32s^4}$

These equations are derived on the basis that θ is small. θ and p are obtained from the relationships

$$p - n - \tfrac{1}{2} = s \tag{9}$$

$$2z = s + \theta \tag{10}$$

By restricting the value of θ to $|\theta| \leqq 0.5$, s is forced to take on the half integral value nearest to $2z$, since both p and n must be integral. Using this method the best results were obtained by calculating $K_2(z)$ and $K_3(z)$ and using the recursion formula

$$K_{n+1}(z) = K_{n-1}(z) + \frac{2n}{z} K_n(z) \tag{11}$$

to calculate the $K_n(z)$ required. In the program for calculating $K_n(z)$ a convergence test was applied to each term of the series in Eq (3); the summation was terminated at the *least* term and, since the terms alternate in sign, one half of the following term was added to the sum.

An alternative treatment due to Burnett [4] starts from a representation of R_p/u_p as a double integral which is subsequently reduced to a series of products of gamma and hypergeometric functions. Expanding this series leads to the following expression for the converging factor, which in this case is applied to the term following the *least* term of the series for $K_n(z)$

$$\begin{aligned}\frac{R_p}{u_p} = \frac{1}{2} &+ \frac{1}{8z}[\sigma - n + 1] + \frac{1}{32z^2}\left[\left(\frac{1}{2} - n\right)(2\sigma - 3n) + \sigma^2 + \frac{1}{2}\sigma - \frac{1}{4}\right] \\ &+ \frac{1}{128z^3}\left[\left(\frac{1}{2} - n\right)\left(3\sigma^2 - 11\sigma n + 9n^2 - \frac{5}{2}\sigma - \frac{5}{2}\right) + \sigma^3 - \frac{5}{4}\sigma - \frac{1}{8}\right] \\ &+ \frac{1}{512z^4}\left[\left(\frac{1}{2} - n\right)\left(4\sigma^3 - 28\sigma^2 n + 48\sigma n^2 - 27n^3 - 11\sigma^2 + 11\sigma n - \frac{5}{2}n^2\right.\right. \\ &\qquad \left.\left. - 11\sigma + 16n + \frac{27}{8}\right) + \sigma^4 - \sigma^3 - \frac{11}{4}\sigma^2 + \frac{7}{8}\sigma + \frac{13}{16}\right]\end{aligned} \tag{12}$$

where

$$\sigma = 2z - p + n - \tfrac{1}{2}. \tag{13}$$

Previously published versions of this series [4], [5] go only as far as the term in z^{-2}. It is easy to show that the successive powers of z^{-1} given in Eq (12) may be derived completely from the first five terms of Eq (3); in other words, to the extent that the equations appropriate to them have been given, the two methods should be roughly

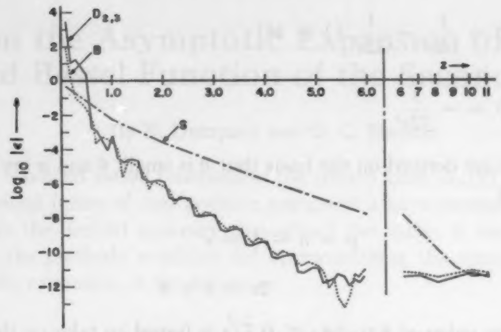

FIG. 1.—Deviation plot for calculations of $K_0(z)$

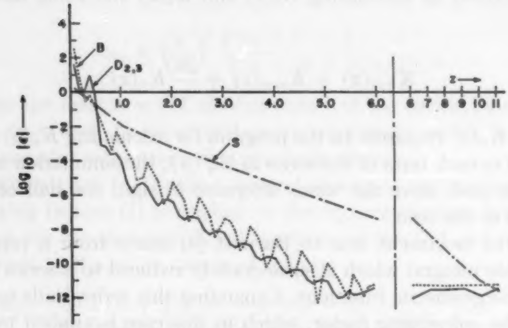

FIG. 2.—Deviation plot for calculations of $K_1(z)$

equivalent. In applying the Burnett treatment the converging factors for K_0 and K_1 were calculated from Eq (12) and functions of higher order obtained by use of the recursion formula.

The accompanying figures show graphs of $\log_{10}|\epsilon|$ against the argument z for $K_0(z)$ and $K_1(z)$ calculated using the Dingle ($D_{2,3}$) and Burnett (B) methods, where

$$\epsilon = \frac{K(\text{exact}) - K(\text{calc.})}{K(\text{exact})}. \tag{14}$$

The values for K(exact) were taken from a paper by Aldis [6] which gives values of $K_0(z)$ and $K_1(z)$ in general accurate to more places than are required here. Curves S were obtained using Eq (1) with R_p/u_p set equal to 0.5. The parts of the curves below about $|\epsilon| = 10^{-11}$ should be ignored since they are influenced by the fact that the computer (a Bendix G15D) carried only twelve figures in the double precision floating point mode used.

It is apparent from Figures 1 and 2 that the Dingle method is somewhat better than that of Burnett. This may be due in part to the fact that the Dingle equations, as given here, contain an infinite number of terms which may be looked on as being

incomplete Burnett type coefficients of powers of z^{-1} higher than the fourth. Above about $z = 10$ it is also evident that for a computer carrying only twelve figures there is nothing to be gained in using a more elaborate converging factor than $R_p/u_p = 0.5$.

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On the Factors of Certain Mersenne Numbers

By John Brillhart and G. D. Johnson

1. Introduction. For the past 10 months the authors have been conducting a search for factors of certain Mersenne numbers on the IBM 701 at the Computer Center, University of California, Berkeley. The following is a report on the nature and results of that search.

2. Extent. Prime factors q were sought for the numbers $M_p = 2^p - 1$ for primes $p < 1200$ in the intervals indicated:

$p = 101$		$2^{30} < q < 2^{35}$
$103 \leqq p \leqq 157,$	$p \neq 151$	$2^{30} < q < 2^{31}$
$157 < p \leqq 257$		$1 < q < 2^{31}$
$257 < p \leqq 1021,$	$p \neq 397$	$1 < q < 2^{30}$
$p = 397$		$1 < q < 2^{32}$
$1021 < p < 1200$		$1 < q < 2^{28}$

No factors $<2^{30}$ were examined for $101 \leqq p \leqq 157$, since these had already been investigated [1]. No M_p were examined for $p < 101$ or $p = 151$, since these numbers have presumably been completely factored. Possible factors $<2^{35}$ were

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also investigated for M_{65537}, the Mersenne number whose exponent is the "last" Fermat prime. M_{397} was investigated to 2^{32} in the hope of finding more small factors.

3. Results.

A. Fifty-five new prime factors were discovered, 6 of which for M_p below the traditional "limit" $p = 257$. These factors are given in the accompanying table, and are indicated by *. Also included are all published prime factors, and 6 new ones (indicated by †) of E. Karst, Brigham Young University. Thus, the table is believed to be a complete listing of all prime factors of M_p for $p < 1200$ known at this time. No factor was found for M_{65537}, whose character is still unknown. Since no factor was found to M_{101} below its cube root, it is the product of two primes.

B. All known prime factors of M_n, $n < 10\ 000$, were tested and found correct, with the exception of the two misprints in H. Riesel [2], as noted earlier by J. Selfridge [3]. In addition, all factors were tested for multiplicity, but no new multiple factors appeared. Hence, to date, only a few multiple factors are known for composite exponents n, while none have been found for prime exponents, further supporting the conjecture that none exist.

4. The Program.

A. STRUCTURE. If $d \mid M_p$, then $d \equiv 1 \pmod{2p}$. Also, since 2 is a quadratic residue of M_n, n odd, then $d \equiv \pm 1 \pmod 8$. Thus, the divisors, d, lie among the common terms t_n of these arithmetic sequences.

In production these terms were generated consecutively by the repeated use of an increment table, which had also been constructed to produce no terms divisible by 3, 5, 7, or 11. (See [1].)

Divisibility of M_p by each t_n was tested by examining the remainder of M_p $(\mathrm{mod}\ t_n)$ for 0.

For $101 \leqq p \leqq 223$, M_p was reduced mod t_n by multiple precision division.

Example 1. The remainder of M_{101} mod t_n was computed for each t_n by 3 divisions, until t_n was $>2^{31}$, at which time an initial dividend of 67 binary places could be used. This change, which produced the remainder in only 2 divisions, was actually introduced when t_n was $>2^{28}$ by using a modulus of $2^{\alpha}t_n$, $0 < \alpha \leqq 3$, instead of t_n, the error in the final remainder being removed after the last division by an appropriate number of subtractions of t_n, or multiples of t_n. This device was used consistently in all routines whenever possible.

When the program was first run for $p \geqq 223$, the final remainder was computed by residue methods consisting of successive squarings and doublings of the residue of some initial power of 2, followed by a subtraction of 1. Later it was realized, that in a double register machine like the 701, a residue between the initial and final residue could usually be multiplied by a power of 2 greater than the first without producing an illegal divide condition in the registers. The magnitude of the power that could be used was found to depend on the length of the registers (35 binary places) and the length of t_n.

This discovery decreased the testing time for each t_n by about 30%, but greatly complicated the programming, since from the many possible programs, one had to be chosen that required a minimum number of machine cycles.

Table of Factors

p	*Factors*	*p*	*Factors*
2	3	227	
3	7	229	1504073·20492753*·
5	31	233	1399·135607·622577·
7	127	239	479·1913·5737·176383·134000009*·
11	23·89	241	22000409*·
13	8191	251	503·54217·
17	131071	257	
19	524287	263	23671·
23	47·178481	269	13822297*·
29	233·1103·2089	271	
31	2147483647	277	1121297·
37	223·616318177	281	80929·
41	13367·164511353	283	9623·
43	431·9719·2099863	293	
47	2351·4513·13264529	307	14608903*·85798519*·
53	6361·69431·20394401	311	5344847·
59	179951·3203431780337	313	10960009*·
61	2305843009213693951	317	9511·
67	193707721·761838257287	331	
71	228479·48544121·212885833	337	18199·2806537†·
73	439·2298041·9361973132609	347	
79	2687·202029703·1113491139767	349	
83	167·57912614113275649087721	353	931921·
89	618970019642690137449562111	359	719·855857·778165529*·
97	11447·prime	367	12479·51791041*.
101		373	25569151*·
103		379	
107	prime	383	1440847·
109	745988807·	389	56478911*·
113	3391·23279·65993·1868569 ·1066818132868207	397	2383·6353·50023·53993 ·202471·5877983†·
127	prime	401	
131	263·	409	
137		419	839·
139		421	
149		431	863·3449·36238481*·76859369* ·558062249*·
151	18121·55871·165799·2332951·prime		
157	852133201·	433	
163	150287·704161·110211473*·	439	104110607*·
167	2349023·	443	887·
173	730753·1505447·	449	1256303·
179	359·1433·	457	150327409*·
181	43441·1164193·7648337*·	461	2767·
191	383·	463	11113·3407681†·
193	13821503*·	467	121606801*·
197	7487·	479	33385343*·
199		487	4871·
211	15193·	491	983·7707719†·
223	18287·196687·1466449·2916841·	499	20959·

Table of Factors—*Continued*

p	*Factors*	*p*	*Factors*
503		839	26849·
509	12619129†·	853	
521	prime	857	6857·
523		859	7215601·
541		863	8258911·169382737*·
547	5471·	877	35081·1436527*·
557	3343·21993703*·	881	26431·
563		883	8831·63577*·
569	15854617*·55470673*·	887	16173559*·
571	5711·27409*·	907	1170031·
577	3463·	911	1823·26129303*·
587	554129·2926783*·	919	
593	104369·	929	13007·
599		937	28111·
601	3607·64863527*·	941	7529·
607	prime	947	295130657*·
613		953	343081·
617	59233·	967	23209·549257*·
619	110183·	971	
631		977	867577·1813313*·
641	35897·49999*·	983	
643	3189281·	991	
647		997	
653	78557207*·289837969*·	1009	3454817·
659	1319·	1013	6079·
661		1019	2039·75407*
673	581163767*·	1021	40841·795808241*·
677		1031	2063·435502649*·
683	1367·	1033	196271·36913223*·
691		1039	5080711·
701	796337·2983457*·28812503*·	1049	33569·459463*·
709	216868921*·	1051	3575503·
719	1439·772207*·	1061	
727		1063	
733		1069	
739		1087	10722169*·
743	1487·	1091	87281·
751		1093	43721·111487*·
757	9815263·561595591*·	1097	980719·4666639*·
761	4567·6089*·	1103	2207·
769		1109	
773	6864241·9461521†·	1117	53617·
787		1123	
797		1129	33871·
809		1151	
811	326023·	1153	267497·
821	419273207*·	1163	
823		1171	
827	66161·	1181	4742897·
829	72953·	1187	256393·113603023*·
		1193	121687·

In some cases, the initial residue was produced from a comparatively small power of 2 by a single division, while in others, it was obtained from a fairly large power of 2 by multiple-precision division.

Example 2. For M_{397}, 4 different programs were used, each improving on and replacing the preceding, when the length of t_n permitted. The first divisor used was $t_1 = 3 \cdot 794 + 1 = 2383$, which also happens to be the first factor. This is shown below by the calculation schemes of the 4 programs, although only the first was actually used to test such a small possible divisor. With each scheme is also given the interval of t_n, for which it was used. The letters *ir* after a residue indicate the initial residue used by the squaring part of the routine.

I: $1 < t_n < 2^{15}$.	II: $2^{15} < t_n < 2^{17}$.	III: $2^{17} < t_n < 2^{19}$.	IV: $2^{19} < t_n < 2^{21}$.
$2^{35} \equiv 1792 \pmod{2383}$	$2^{60} \equiv 1657 \pmod{2383}$	$2^{62} \equiv 1862 \pmod{2383}$	$2^{64} \equiv 299 \pmod{2383}$
$2^{60} \equiv 1657$	$2^{95} \equiv 342$ *ir*	$2^{97} \equiv 1368$ *ir*	$2^{99} \equiv 706$ *ir*
$2^{95} \equiv 342$ *ir*	$2^{190} \equiv 197$	$2^{194} \equiv 769$	$2^{198} \equiv 389$
$2^{190} \equiv 197$	$2^{196} \equiv 693$	$2^{197} \equiv 1386$	$2^{396} \equiv 1192$
$2^{196} \equiv 693$	$2^{392} \equiv 1266$	$2^{394} \equiv 298$	$2^{397} \equiv 1$
$2^{392} \equiv 1266$	$2^{397} \equiv 1$	$2^{397} \equiv 1$	
$2^{397} \equiv 1$			

B. PRODUCTION. The program was run for 60 hours, each $p < 223$ requiring approximately 23 minutes, and each $p \geqq 223$ requiring from 8 to 18 minutes, the larger exponents taking progressively less time. The special number M_{101} was run for 10 hours.

The routines used are believed to have been accurate, a fact which will be ascertained at a future time, when more rapid computers will accomplish in a few minutes, what has now taken many hours.

The authors would like to thank Professors R. M. Robinson and D. H. Lehmer for their suggestions and ideas, and also the staff of the Computer Center for their many courtesies.

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Further Evaluation of Khintchine's Constant

By John W. Wrench, Jr.

In his fundamental investigation of the metric theory of continued fractions Khintchine [1] proved that the limit, as n tends to infinity, of the geometric mean of the first n partial quotients in the simple continued fraction expansion of almost all real numbers is the absolute constant

$$K = \prod_{r=1}^{\infty} \left(1 + \frac{1}{r(r+2)}\right)^{\ln r/\ln 2}.$$

A different proof, by C. Ryll-Nardzewski, has been recently reproduced by M. Kac [2].

The numerical evaluation of Khintchine's constant was considered by D. H. Lehmer [3]. In addition to finding an approximation to K to 6 decimal places, whose accuracy was subsequently discussed by D. Shanks [4], Lehmer investigated the geometric mean of the first one hundred partial quotients of π.

Recently R. S. Lehman [5] computed the first 1986 partial quotients of π on ORDVAC in order to test the applicability of a similar theorem of Lévy [6], which asserts that, as n tends to infinity, the nth root of the denominator of the nth convergent tends to $\exp(\pi^2/12 \ln 2)$.

Shanks and the writer [7] have studied the representation of K by infinite series and by definite integrals. The computational effectiveness of these series was illustrated by the evaluation of K to 65 decimal places. This calculation has now been extended by me to 155 places, using the same series as previously, namely:

$$\ln 2 \ln K = \ln \frac{3}{2} + \ln 2 \ln \frac{3}{2} - \left\{\frac{1}{2.3} \sum_{k=2}^{\infty} \frac{S_{2k}''}{k} + \frac{1}{4.5} \sum_{k=3}^{\infty} \frac{S_{2k}''}{k} + \frac{1}{6.7} \sum_{k=4}^{\infty} \frac{S_{2k}''}{k} + \cdots\right\},$$

where S_{2k}'' represents

$$\sum_{n=3}^{\infty} n^{-2k} = \zeta(2k) - 1 - 2^{-2k}.$$

A preliminary step in this calculation consisted of the formation of a table of $\zeta(2k)$ to at least 155D for $k = 1(1)\,257$. The first 60 entries of this table were computed by the formula

$$\zeta(2k) = (-1)^{k-1} \frac{B_{2k}\,(2\pi)^{2k}}{2(2k)!},$$

where the notation for the Bernoulli numbers is that used by K. Knopp [8]. The numerical values of these numbers were taken from the tables of H. T. Davis [9]. The requisite decimal approximations to $\pi^{2k}/(2k)!$ were obtained from my manuscript table [10] of such data. The remaining entries were computed directly from the series defining $\zeta(2k)$, a maximum of eighteen terms being required initially.

From these values of $\zeta(2k)$ the approximations to S_{2k}'' and S_{2k}''/k were then computed to 155D. All these data were subjected to the following check relations:

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$$\sum_{k=1}^{\infty} [\zeta(2k) - 1] = \frac{3}{4},$$

$$\sum_{k=1}^{\infty} S_{2k}'' = \frac{5}{12},$$

$$\sum_{k=1}^{\infty} S_{2k}''/k = \ln \frac{3}{2},$$

$$\sum_{r=1}^{\infty} \sum_{k=r}^{\infty} S_{2k}''/k = \frac{5}{12}.$$

Substitution of the computed values in these formulas resulted in discrepancies all less than 3 units in the 155th decimal place.

The final results of this calculation when rounded to 155D are as follows:

$\ln 2 \ln K =$

0.68472 47885 63157 12329 91461 48755 77762 04606 75416 33744
88366 06289 86781 59568 82176 26936 10437 07681 43495 85810
09970 15696 93974 12470 41578 92227 14396 39612 78766 18097
72947 $\cdots$,

$\ln K =$

0.98784 90568 33810 78966 92547 27147 07295 43261 99254 96088
67354 27755 30068 72109 27094 18512 90938 20768 83372 75259
67479 51231 68801 78544 35925 75519 06227 59695 60965 06769
43483 $\cdots$,

$K =$

2.68545 20010 65306 44530 97148 35481 79569 38203 82293 99446
29530 51152 34555 72188 59537 15200 28011 41174 93184 76979
95153 46590 52880 90082 89767 77164 10963 05179 25334 83259
66838 $\cdots$.

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Note on $\int_0^\infty e^{-x} J_0\left(\frac{\eta x}{\xi}\right) J_1\left(\frac{x}{\xi}\right) x^{-n}\,dx$

By Henry E. Fettis

In a recent paper this integral has been calculated numerically by Weeg [1] for the cases $n = 0, 1$. The following analytic expression for this integral when $n = 0$ in terms of elliptic integrals is given by Byrd and Friedman ([2], formula 563.01):

$$
\begin{aligned}
\bar{I}(p, r, s) &= \int_0^\infty e^{-pt} J_0(rt)\, J_1(st)\, dt = \frac{1}{s}\left[1 - \Lambda_0(\beta, k)\right] \\
&= \frac{1}{s}\left[1 - \frac{2}{\pi}\left\{E(k)\, F(\beta, k') + K(k)\left[E(\beta, k') - F(\beta, k')\right]\right\}\right],
\end{aligned} \tag{1}
$$

with

$$
k^2 = \frac{[N - p^2 - r^2 + s^2]\,[N - p^2 + r^2 - s^2]}{[N + p^2 + r^2 - s^2]\,[N + p^2 - r^2 + s^2]},
$$

$$
\sin^2\beta = \frac{1}{2}\left[1 + \frac{p^2 + r^2 + s^2}{N}\right],
$$

$$
N = \sqrt{(p^2 + r^2 + s^2)^2 + 4p^2s^2}.
$$

$\Lambda_0(\beta, k)$ is known as Heuman's Lambda Function.

Calculations based on the above expressions do not agree with the ones made numerically in [1], and since the latter values have been verified by an independent relationship given in Weeg's paper,* the author was led to attempt to verify the above analytic expression independently. A closer examination of Eq. (1) leads one to question its validity by consideration of various special cases for which simpler expressions can be found. For example, it is known ([3], Chapter III, Art. 7) that

$$
\bar{I}(p, o, s) = \frac{\sqrt{p^2 + s^2} - p}{s\sqrt{p^2 + s^2}}, \tag{2}
$$

whereas (1) fails to reduce to this simple form when $r = 0$. It is also possible to demonstrate directly the relations:

$$
s\bar{I}(p, r, s) + r\,\bar{I}(p, s, r) = 1 - \frac{2pK(k)}{\pi\sqrt{p^2 + (r + s)^2}} = 1 - \frac{p}{\pi\sqrt{rs}}\,Q_{-\frac{1}{2}}(x) \tag{3}
$$

where

$$
k^2 = 4rs/[p^2 + (r + s)^2], \qquad x = (p^2 + r^2 + s^2)/2rs
$$

and

$$
\begin{aligned}
\bar{I}(0, r, s) &= 1/s, && r < s, \\
&= 1/2s, && r = s, \\
&= 0, && r > s,
\end{aligned} \tag{4}
$$

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* Eq. 2 of [1] should read $I(\eta, \xi, 0) + I(\eta, \xi, 1) = \frac{1}{\pi\sqrt{\eta}}\,[Q_{-\frac{1}{2}}(x) - \eta Q_{\frac{1}{2}}(x)]$.

neither of which can be obtained as a special case of (1). In fact, for $p = 0$, the right side of (1) reduces identically to zero, irrespective of the relative magnitudes of r and s. A simpler expression for this integral has been derived by the present author in terms of complete elliptic integrals of the first and third kinds, by making use of the representation ([3], Chap. III, Art. 6):

$$\pi J_0(rt)J_1(st) = \int_0^\pi J_0[t\sqrt{r^2 + s^2 - 2rs\cos\phi}]\, d\phi. \tag{5}$$

Differentiation of (5) gives

$$\pi J_0(rt)J_1(st) = \int_0^\pi J_1[t\sqrt{r^2 + s^2 - 2rs\cos\phi}]\, \frac{(s - r\cos\phi)\, d\phi}{\sqrt{r^2 + s^2 - 2rs\cos\phi}}. \tag{6}$$

Thus

$$\begin{aligned}\pi \int_0^\infty e^{-pt} J_0(rt)J_1(st)\, dt &= \int_0^\infty e^{-pt}\, dt \int_0^\pi J_1[t\sqrt{r^2 + s^2 - 2rs\cos\phi}] \\ &\qquad \cdot \frac{(s - r\cos\phi)\, d\phi}{\sqrt{r^2 + s^2 - 2rs\cos\phi}} \\ &= \int_0^\pi \frac{(s - r\cos\phi)\, d\phi}{\sqrt{r^2 + s^2 - 2rs\cos\phi}} \int_0^\infty e^{-pt} J_1[t\sqrt{r^2 + s^2 - 2rs\cos\phi}]\, dt.\end{aligned} \tag{7}$$

Carrying out the integration with respect to "t" gives

$$\begin{aligned}\pi \int_0^\infty e^{-pt} J_0(rt)J_1(st)\, dt = \int_0^\pi &\frac{(s - r\cos\phi)}{r^2 + s^2 - 2rs\cos\phi} \\ &\left[1 - \frac{p}{\sqrt{p^2 + r^2 + s^2 - 2rs\cos\phi}}\right] d\phi.\end{aligned} \tag{8}$$

The first integral on the right side of (8) is known ([4], formula 860.2):

$$\begin{aligned}\int_0^\pi \frac{[s - r\cos\phi]\, d\phi}{r^2 + s^2 - 2rs\cos\phi} &= \pi/s, & r &< s, \\ &= \pi/2s, & r &= s, \\ &= 0, & r &> s,\end{aligned} \tag{9}$$

and the second can be converted to elliptic integrals in standard form by substituting $\phi = \pi - 2\theta$ and by rearranging, thus,

$$\begin{aligned}\int_0^\pi \frac{(s - r\cos\phi)\, d\phi}{(r^2 + s^2 - 2rs\cos\phi)\sqrt{p^2 + r^2 + s^2 - 2rs\cos\phi}} &= \frac{1}{s\sqrt{p^2 + (r + s)^2}} \\ \cdot \left\{\frac{s - r}{s + r} \int_0^{\pi/2} \frac{d\theta}{(1 - \alpha^2\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} + \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}\right\} \\ &= \frac{1}{s\sqrt{p^2 + (r + s)^2}} \left\{\frac{s - r}{s + r} \prod(\alpha, k) + K(k)\right\},\end{aligned} \tag{10}$$

where

$$\alpha^2 = \frac{4rs}{(r + s)^2},$$

$$k^2 = \frac{4rs}{p^2 + (r + s)^2},$$

and $K(k)$ and $\prod(\alpha, k)$ are complete elliptic integrals of the first and third kinds, respectively. An alternative form which expresses the result in terms of Heuman's Lambda Function can be obtained from formula 413.01 of [2]:

$$(11)\quad \begin{aligned} p \int_0^\pi &\frac{(s - r\cos\phi)\,d\phi}{(r^2 + s^2 - 2rs\cos\phi)\sqrt{p^2 + r^2 + s^2 - 2rs\cos\phi}} \\ &= \frac{1}{s}\left\{\frac{\pi}{2}\frac{|s - r|}{(s - r)}\Lambda_0(\beta, k) + \frac{pK(k)}{\sqrt{p^2 + (r + s)^2}}\right\}, \end{aligned}$$

with

$$\sin\beta = \frac{p}{\sqrt{p^2 + (r - s)^2}}.$$

The final result is

$$(12)\quad \begin{aligned} \int_0^\infty e^{-pt} J_0(rt) J_1(st)\,dt &= \frac{1}{s}\left[1 - \frac{1}{2}\Lambda_0(\beta, k) - \frac{pK(k)}{\pi\sqrt{p^2 + (r + s)^2}}\right], & r < s, \\ &= \frac{1}{s}\left[-\frac{1}{2}\frac{pK(k)}{\pi\sqrt{p^2 + 4r^2}}\right] & r = s, \\ &= \frac{1}{s}\left[\frac{1}{2}\Lambda_0(\beta, k) - \frac{pK(k)}{\pi\sqrt{p^2 + (r + s)^2}}\right], & r > s, \end{aligned}$$

where

$$k^2 = \frac{4rs}{p^2 + (r + s)^2}, \qquad \sin\beta = \frac{p}{\sqrt{p + (r + s)^2}}.$$

Equation (12) when $r = s$, incidentally, gives a particularly simple expression for Weeg's integral in those cases where $\eta = 1$.

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1. Gerald P. Weeg, "Numerical Integration of $\int_0^\infty e^{-x} J_0\left(\frac{\eta x}{\xi}\right) J_1\left(\frac{x}{\xi}\right) x^{-n}\,dx$," *MTAC*, v. 13, 1959, p. 312–313.

2. Paul F. Byrd & Morris D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer, Berlin, 1954.

3. W. Magnus & F. Oberhettinger, *Formeln und Satze fur die Speziellen Funktionen der Mathematischen Physik*, 2nd Edition, Springer, Berlin, 1948.

4. H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, Third Edition, Macmillan, New York, 1957.

Integrals of Products of Laguerre Polynomials

By R. D. Lord

In a recent note in this journal Gillis and Weiss [1] have evaluated as a finite sum, and found a recurrence relation for, the integral

$$C_{rst} = \int_0^\infty e^{-x} L_r(x) L_s(x) L_t(x)\, dx, \tag{1}$$

which gives the coefficients in the expansion

$$L_r(x) L_s(x) = \sum C_{rst} L_t(x),$$

where $L_n(x)$ denotes the Laguerre polynomial,

$$L_n(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x^\nu/\nu! \tag{2}$$

This attracted my interest since about ten years ago I considered the more general integral of the product of generalized Laguerre polynomials

$$\int_0^\infty e^{-x} x^\beta L_{n_1}^{(\alpha_1)}(a_1 x) L_{n_2}^{(\alpha_2)}(a_2 x) \cdots L_{n_m}^{(\alpha_m)}(a_m x)\, dx, \tag{3}$$

with the object of finding cases which simplified. I tried methods which I had used with the corresponding problem for Hermite polynomials [2], but met with little success. But the special case (1) is easily dealt with as follows.

The generating function for Laguerre polynomials is

$$\sum_{r=0}^{\infty} (-1)^r L_r(x)\, u^r = \frac{1}{1+u} \exp\left(\frac{xu}{1+u}\right), \tag{4}$$

and hence that for the product of three polynomials is

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^{r+s+t} L_r(x) L_s(x) L_t(x)\, u_1^r u_2^s u_3^t$$

$$= [(1+u_1)(1+u_2)(1+u_3)]^{-1} \exp x\left(\frac{u_1}{1+u_1} + \frac{u_2}{1+u_2} + \frac{u_3}{1+u_3}\right).$$

On multiplying by e^{-x} and integrating from 0 to ∞, we get

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^{r+s+t} C_{rst} u_1^r u_2^s u_3^t = (1 - u_2 u_3 - u_3 u_1 - u_1 u_2 - 2u_1 u_2 u_3)^{-1} \tag{5}$$

as the generating function for C_{rst}. If we write the right hand side of (5) as

$$\sum (u_2 u_3 + u_3 u_1 + u_1 u_2 + 2u_1 u_2 u_3)^\nu,$$

we immediately see that C_{rst} is an integer with the sign $(-)^{r+s+t}$. Further expansion of each term by the multinomial theorem gives Watson's formula [3]

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$$(6) \qquad C_{rst} = (-2)^{r+s+t} \sum_n 2^{-2n} \frac{n!}{(n-r)!(n-s)!(n-t)!(r+s+t-2n)!},$$

where in the summation all factorials have non-negative arguments.

If we transpose the right hand side of (5) and equate coefficients of $u_1^r u_2^s u_3^t$, we have

$$(7) \qquad C_{rst} = C_{r,s-1,t-1} + C_{r-1,s,t-1} + C_{r-1,s-1,t} + 2C_{r-1,s-1,t-1},$$

which is the recurrence formula with simplest coefficients, especially as the right hand side can be treated as the sum of five terms. Even though all suffixes vary, it probably provides the quickest way of computing all values of C_{rst} up to a given set of r, s, t. For some machines at least, it may well give the quickest way for calculating a given C_{rst}, and it provides an easy method for desk-machine computation when r, s and t are small. For computations by other methods it provides a simple check. Other checks may be obtained by giving u_1 special values in (5) and equating coefficients of $u_2^s u_3^t$. Putting $u_1 = -1$, we get

$$(8) \qquad \sum_r C_{rst} = 1,$$

given by Gillis and Weiss; putting $u_1 = -\frac{1}{2}$, we get

$$(9) \qquad \sum_r 2^{s+t-r} C_{rst} = \binom{s+t}{s}.$$

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1. J. Gillis & G. Weiss, "Products of Laguerre polynomials," *Math. Comp. (MTAC)*, v. 14, 1960, p. 60–63.
2. R. D. Lord, "Some integrals involving Hermite polynomials," London Math. Soc., *Jn.*, v. 24, 1949, p. 101–112.
3. G. N. Watson, "A note on the polynomials of Hermite and Laguerre," London Math. Soc., *Jn.*, v. 13, 1938, p. 29.

The Evaluation of Certain Probability Integrals

By Irwin Greenberg

A problem which often arises in the field of probability and statistics is the following:

Assume that there are n independent stochastic processes and that the kth process has an output distribution $f_k(x_k)$. The probability that the jth process yields a higher output than any of the others is

$$(1) \qquad P_j = \int_{-\infty}^{\infty} f_j(x_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} \int_{-\infty}^{x_j} f_k(x_k)\, dx_k\, dx_j.$$

In certain special cases, equation (1) is easily integrated; for example, if the various $f_k(x_k)$ are all uniform or exponential distributions.

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In certain other cases, equation (1) can be put into a form which is readily evaluated by hand computation, using well known statistical reference tables. This paper will illustrate one of these cases; for $n = 2$ or 3 and the x_k having a normal distribution.

Under the assumption of normality,

$$f_k(x_k) = (\sqrt{2\pi}\,\sigma_k)^{-1} \exp\left[-\frac{1}{2}\left(\frac{x_k - \mu_k}{\sigma_k}\right)^2\right]; \qquad k = 1,2,3 \tag{2}$$

where μ_k and σ_k are the mean and standard deviation, respectively. Letting

$$X_k = \frac{x_k - \mu_1}{\sigma_1}, \tag{3}$$

equation (2) becomes

$$f_k(X_k) = (\sqrt{2\pi}\,\sigma_k')^{-1} \exp\left[-\frac{1}{2}\left(\frac{X_k - \mu_k'}{\sigma_k'}\right)^2\right] \tag{4}$$

where

$$\mu_k' = \frac{\mu_k - \mu_1}{\sigma_1}; \qquad \sigma_k' = \frac{\sigma_k}{\sigma_1}. \tag{5}$$

The probability that the output of process 1 exceeds the output of process 2 is

$$\begin{aligned} P_1 &= (2\pi\sigma_2')^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{X_1} \exp\left[-\frac{1}{2}X_1^2 - \frac{1}{2}\left(\frac{X_2 - \mu_2'}{\sigma_2'}\right)^2\right] dX_2\, dX_1 \\ &= (\sqrt{2\pi})^{-1} \int_{-\infty}^{M_2} \exp\left(-\frac{t^2}{2}\right) dt, \end{aligned} \tag{6}$$

where

$$M_2 = -\mu_2'/\sqrt{(\sigma_2')^2 + 1}. \tag{7}$$

The simplification of equation (6) is obtained by expanding the exponent, grouping terms, completing the square in X_1 and reversing the order of integration after making the transformations:

$$X_1 = X_1';\ X_2 = X_1' + X_2' + \mu_2'. \tag{8}$$

Equation (6) is the cumulative normal function and is tabulated in most texts on statistics. In a similar manner, it can be shown that the probability that the output of process 1 exceeds the output of process 2 and of process 3 is

$$\begin{aligned} P_1 &= [(2\pi)^{3/2}\sigma_2'\,\sigma_3']^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{X_1} \int_{-\infty}^{X_1} \exp\left[-\frac{1}{2}X_1^2 - \frac{1}{2}\left(\frac{X_2 - \mu_2'}{\sigma_2'}\right)^2 \right. \\ &\qquad \left. - \frac{1}{2}\left(\frac{X_3 - \mu_3'}{\sigma_3'}\right)^2\right] dX_3\, dX_2\, dX_1 \\ &= \left[2\pi\sqrt{1 - r^2}\right]^{-1} \int_{-\infty}^{M_2} \int_{-\infty}^{M_3} \exp\left[-\frac{1}{2(1 - r^2)}(t_2^2 - 2rt_2 t_3 \right. \\ &\qquad \left. + t_3^2)\right] dt_3\, dt_2, \end{aligned} \tag{9}$$

where

$$
(10) \quad \begin{cases} M_2 = -\mu_2'/\sqrt{(\sigma_2')^2 + 1} \\ M_3 = -\mu_3'/\sqrt{(\sigma_3')^2 + 1} \\ \dfrac{1}{r} = \sqrt{(\sigma_3')^2 + 1} \quad \sqrt{(\sigma_2')^2 + 1} \, . \end{cases}
$$

Equation (10) gives the volume under the bivariate normal probability surface with correlation coefficient r. These volumes are tabulated in [1].

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1. NBS Applied Mathematics Series, No. 50, *Tables of the Bivariate Normal Distribution Function and Related Functions*, U. S. Government Printing Office, Washington, D. C. 1959.

The Congruence $2^{p-1} \equiv 1 \pmod{p^2}$ for $p < 100{,}000$

By Sidney Kravitz

Fröberg has previously announced [1] the computation of the Fermat remainders corresponding to all odd primes less than 50,000. His results show that $p = 1093$ and $p = 3511$ are the only solutions of the congruence $2^{p-1} \equiv 1 \pmod{p^2}$ in that range.

The residues of $2^{p-1} \pmod{p^2}$ have been computed for $50{,}000 < p < 100{,}000$ on an IBM 650 system at Picatinny Arsenal. No residue congruent to 1 was found corresponding to a prime in this range.

A copy of the table of residues has been deposited in the Unpublished Mathematical Tables file.

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1. C. E. Fröberg, "Some Computations of Wilson and Fermat Remainders," *MTAC*, v. 12, 1958, p. 281.

Editorial Note: Reference should also be made to:

1. W. Meissner, "Uber die Teilbarkeit von $2^p - 2$ durch das Quadrat der Primzahl $p = 1093$," Akad. d. Wiss, Berlin, *Sitzungsb.*, v. 35, 1913, p. 663–667
2. N. G. W. H. Beeger, "On a new case of the congruence $2^{p-1} \equiv 1 \pmod{p^2}$," *Messenger Math.*, v. 51, 1922, p. 149–150.

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Values of $\frac{2}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^n dt$

By Kasaburô Harumi,† Shigetoshi Katsura,† and John W. Wrench, Jr.‡

In discussing the equation of state for the molecules of one-dimensional square well potential [1], the first two authors required the numerical values of the integral

$$I_n = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^n dt = \frac{1}{2^{n-1}\Gamma(n)} \sum_{p=0}^{[(n-1)/2]} (-1)^p \binom{n}{p} (n-2p)^{n-1}.$$

Inasmuch as these values seem to be of use in other applications, and apparently have not been previously tabulated, the first two authors calculated on SENAC-1 (Sendai Automatic Computer 1) a six-place table of I_n for $n = 1(1)\ 30$.

The last author reviewed this table and recalculated the data, finding the corresponding exact rational values.

The authors have decided to publish the ten-place table of I_n which is appended.

TABLE OF $I_n \equiv \frac{2}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^n dt$

n	I_n	n	I_n
1	1.00000 00000	16	0.34224 02614
2	1.00000 00000	17	0.33220 82691
3	0.75000 00000	18	0.32300 93942
4	0.66666 66667	19	0.31453 44009
5	0.59895 83333	20	0.30669 31017
6	0.55000 00000	21	0.29941 02903
7	0.51102 43056	22	0.29262 26872
8	0.47936 50794	23	0.28627 66141
9	0.45292 09682	24	0.28032 61985
10	0.43041 77690	25	0.27473 19735
11	0.41096 26428	26	0.26945 97712
12	0.39392 55652	27	0.26447 98425
13	0.37884 40845	28	0.25976 61480
14	0.36537 08695	29	0.25529 57845
15	0.35323 91567	30	0.25104 85320

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1. S. Katsura & K. Harumi, "A note on the Born-Green linearized integral equation," Phys. Soc. of London, *Proc.*, v. 75, 1960, p. 826–832.

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Computation of Fresnel Integrals

By J. Boersma

Two approximations, one valid for x less than 4 and the other valid for x larger than 4, have been established by means of the τ-method of Lanczos [1] for the Fresnel integrals defined in the form

$$f(x) = \int_0^x \frac{e^{-it}}{\sqrt{2\pi t}}\, dt = C(x) - iS(x).$$

These approximations are the following:

$$\text{(1)} \quad \text{For } 0 \leqq x \leqq 4 \qquad f(x) = e^{-ix} \sqrt{\frac{x}{4}} \sum_{n=0}^{11} (a_n + ib_n) \left(\frac{x}{4}\right)^n$$

$$\text{(2)} \quad \text{For } x \geqq 4 \qquad f(x) = \frac{1-i}{2} + e^{-ix} \sqrt{\frac{4}{x}} \sum_{n=0}^{11} (c_n + id_n) \left(\frac{4}{x}\right)^n.$$

The numerical values of the coefficients a_n, b_n, c_n and d_n are given by

$a_0 = +1.595769140$	$b_0 = -0.000000033$	$c_0 = 0$	$d_0 = +0.199471140$
$a_1 = -0.000001702$	$b_1 = +4.255387524$	$c_1 = -0.024933975$	$d_1 = +0.000000023$
$a_2 = -6.808568854$	$b_2 = -0.000092810$	$c_2 = +0.000003936$	$d_2 = -0.009351341$
$a_3 = -0.000576361$	$b_3 = -7.780020400$	$c_3 = +0.005770956$	$d_3 = +0.000023006$
$a_4 = +6.920691902$	$b_4 = -0.009520895$	$c_4 = +0.000689892$	$d_4 = +0.004851466$
$a_5 = -0.016898657$	$b_5 = +5.075161298$	$c_5 = -0.009497136$	$d_5 = +0.001903218$
$a_6 = -3.050485660$	$b_6 = -0.138341947$	$c_6 = +0.011948809$	$d_6 = -0.017122914$
$a_7 = -0.075752419$	$b_7 = -1.363729124$	$c_7 = -0.006748873$	$d_7 = +0.029064067$
$a_8 = +0.850663781$	$b_8 = -0.403349276$	$c_8 = +0.000246420$	$d_8 = -0.027928955$
$a_9 = -0.025639041$	$b_9 = +0.702222016$	$c_9 = +0.002102967$	$d_9 = +0.016497308$
$a_{10} = -0.150230960$	$b_{10} = -0.216195929$	$c_{10} = -0.001217930$	$d_{10} = -0.005598515$
$a_{11} = +0.034404779$	$b_{11} = +0.019547031$	$c_{11} = +0.000233939$	$d_{11} = +0.000838386$

The derivation of these approximations is given in [2].

The maximum error is 1.6×10^{-9} for the first approximation and 0.5×10^{-9} for the second approximation.

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1. C. Lanczos, *Applied Analysis*, Prentice Hall, Englewood Cliffs, N. J., 1956.
2. J. Boersma, "On a numerical method for the computation of Fresnel integrals", *Report TW 2*, Math. Inst., Univ. of Groningen, 1960.

Received March 2, 1960.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

69[A, J, L, M].—I. M. Ryshik & I. S. Gradstein, *Summen-Produkt- und Integral-Tafeln: Tables of Series, Products, and Integrals*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1957, xxiii + 438 p., 27 cm. Price DM56.

This volume of tables consists of a translation of the Russian third edition [1] into parallel German and English text. That edition has now been improved and augmented by the incorporation of corrections listed on a sheet of corrigenda accompanying the third edition, the addition of supplementary remarks in the Appendix, and the inclusion of an extensive supplementary bibliography, which consists of books and monographs pertaining to integral transforms, special functions, and indexes of mathematical tables.

The preparation of the third edition was carried out by I. S. Gradstein, following the death in the Second World War of I. M. Ryshik, who was responsible for the first two editions. A description of the contents of the first edition has appeared in a detailed review by R. C. Archibald [2].

The present book represents an extensive revision of the earlier editions. Major changes include the deletion of sections relating to the calculus of finite differences (including formulas for numerical quadrature), the addition of a completely new chapter on integral transforms, and the enlargement of the chapters on special functions.

Information on each of the special functions—in particular, elliptic, cylindrical, and spherical—is presented systematically. Such information generally includes definitions; representation by integrals, series, and products; asymptotic formulas; functional equations; special values; and theorems relating to characteristic properties.

An introductory section entitled "On the Arrangement of the Formulae" explains the arrangement of the contents of the chapters on elementary functions and on their definite integrals, and sets forth innovations in the arrangement of definite integrals, which in previous editions followed closely the classification established in the classical tables of Bierens de Haan [3].

The usefulness of this volume is enhanced by references and cross-references for the sources of most of the 5400 formulas presented. Formulas are numbered decimally within each chapter, and the chapter numbers are used for the integer part, as is customary. Furthermore, a key for the references to the literature cited on p. 434 is described in the Preface. It seems appropriate to note here that in both the Russian third edition and in this translation the list of numbered references consists of 40 items, although reference is made in several places in the book to a forty-first item and a forty-second that were apparently omitted inadvertently.

The first two editions contained a table of 10D approximations to $(2n - 1)!!/(2n)!!$ and $(2n - 1)!!/(2n)!!(2n + 1)$, for $n = 1(1)15$, and to $(2n - 1)!!/(2n + 2)!!$ and $(2n - 1)!!/(2n + 2)!!(2n + 3)$, for $n = 1(1)14$. This numerical information is now supplemented by an original table of the Lobatschefsky function $L(x)$ to 7D for $x = 0°(1°)10°$, 6D for $x = 11°(1°)30°$, and 5D for $x = 31°(1°)90°$, computed by N. V. Tomantovaia͡ under the supervision of B. L. Laptev. This function

is briefly discussed (p. 296–297) in the chapter on special functions. Additional numerical data also include exact values of the first 17 Bernoulli numbers and the first 10 Euler numbers, 10D approximations to $\zeta(n)$ for $n = 2(1)11$, and Euler's constant and Catalan's constant to 16D and 9D, respectively. I have examined all these data carefully, and the errors detected, together with errors in the formulas, are enumerated separately in this issue (MTE 293).

Use of the book is facilitated by an elaborate index of special functions and notations on p. 417–422. In addition to supplementary remarks and the bibliographies already mentioned, the Appendix contains (on p. 423–429) a discussion of the variations in the notation and symbols used for special numbers and functions throughout the mathematical literature and a concise list of abbreviations (p. 432–433).

The lucid expository style employed throughout is exemplified in the Introduction. Here, a systematic summary of definitions and theorems relating to infinite products and infinite series of various types supplements the list of relevant formulas. Similar explanatory text serves as introduction to several of the subsequent chapters and their subdivisions.

Typographical errors found in the text are minor and do not detract from the intelligibility of the textual material. The typography, especially in a compilation of such a large number of formulas, is uniformly excellent, and the appearance of the book is attractive. Professor Archibald's opinion that the first edition was "undoubtedly of considerable value for any mathematician to have at hand" certainly holds true for this latest version.

J. W. W.

1. I. M. Ryzhik & I. S. Gradshteĭn, *Tablitsy Integralov, Summ, Riadov i Proizvedeniĭ*, [Tables of Integrals, Sums, Series and Products], The State Publishing House for Technical and Theoretical Literature, Moscow, 1951.
2. R. C. Archibald, **RMT** 219, *MTAC*, v. 1, 1943/45, p. 442.
3. Bierens de Haan, *Nouvelles Tables d'Intégrales Définies*, Leyden 1867. Reprinted by G. E. Stechert & Co., New York, 1939.

70[G].—Eugene Prange, *An Algorism for Factoring $X^n - 1$ over a Finite Field*, AFCRC-TN-59-775, U. S. Air Force, Bedford, Mass., October 1959, iii + 20 p., 27 cm.

An algorism is given for factoring $X^n - 1$ over the finite field F_q of q elements. This can be of use in constructing another finite field over F_q, in constructing a linear recursion of period n over F_q, or in constructing cyclic error-correcting group codes. The algorism has two parts: Step 1, the construction of the multiplicative identities of the minimal ideals of $F_q[X]/[X^n - 1]$; Step 2, the use of these idempotents in the construction of the irreducible factors of $X^n - 1$.

Author's Abstract

71[G].—M. Rotenberg, R. Bivins, N. Metropolis & J. K. Wooten, Jr., *The 3-j and 6-j Symbols*, The Technology Press, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1960, viii + 498 p., 29 cm. Price $16.00.

Wigner's 3-j symbol is closely related to the Clebsch-Gordan coefficients used in the coupling of angular momenta. If $\mathbf{J}_1$ and $\mathbf{J}_2$ are coupled to give $\mathbf{J}$, with j, j_1, j_2 as the total-angular-momentum quantum numbers and m, m_1, m_2 as the quantum

numbers for the z-components, the expansion coefficient giving the coupled states in terms of the uncoupled are

$$(j_1j_2jm \mid j_1m_1j_2m_2) = (-1)^{j_2-j_1-m}(2j+1)^{1/2}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}.$$

Here the symbol on the left is the expansion coefficient in the notation of Condon and Shortley, *Theory of Atomic Spectra*; the last symbol on the right is the Wigner 3-j symbol. The advantage of a tabulation of the 3-j symbols,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix},$$

rather than of expansion coefficients results from the high degree of symmetry of the 3-j symbols. At most a sign change results from an interchange of columns or from changing the signs of all the m's. Thus, from these tables, which are restricted to

$$j_1 \geqq j_2 \geqq j_3 \quad \text{and} \quad m_2 \leqq 0,$$

all expansion coefficients can be obtained for any $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, 8$.

The 6-j symbols occur in the coupling of three angular momenta $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}_3$. One can either couple $\mathbf{J}_1$ and $\mathbf{J}_2$ first to obtain $\mathbf{J}'$, and then couple $\mathbf{J}'$ to $\mathbf{J}_3$ to obtain $\mathbf{J}$—this coupling scheme results in quantum numbers j_1, j_2, j', j_3, j, m—or one can first couple $\mathbf{J}_2$ and $\mathbf{J}_3$ to obtain $\mathbf{J}''$, and then couple $\mathbf{J}''$ to $\mathbf{J}_1$ to obtain $\mathbf{J}$—this scheme results in quantum numbers j_2, j_3, j'', j_1, j, m. The overlap integral between these two representations is given by

$$(j_1j_2j'j_3jm \mid j_2j_3j''j_1jm) = (-1)^{j_1+j_2+j_3+j}[(2j'+1)(2j''+1)]^{1/2}\begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & j & j'' \end{Bmatrix},$$

where the last symbol is a 6-j symbol. The 6-j symbol tabulated,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix},$$

has sufficient symmetry that it need be listed only for

$$j_1 \geqq j_2 \geqq j_3, \quad j_1 \geqq l_1, \quad j_2 \geqq l_2, \quad j_2 \geqq l_3.$$

Within these restrictions, it is listed for all half-integral values of the j's and l's from 0 to 8.

The tables were computed on the MANIAC II at the Los Alamos Scientific Laboratory. An adequate 40-page introduction describes the symbols and their uses. Since the symbols are the square-roots of rational fractions, the squares are tabulated as powers of primes in a shorthand notation, with an asterisk used to denote the negative square root. For example, the entry $*\underline{1}5\underline{1}0,\underline{2}\underline{2}\underline{2}1$ is to be interpreted as

$$-\left[\frac{3^5 \times 7^0 \times 19^1}{2^1 \times 5^1 \times 11^2 \times 13^2 \times 17^2}\right]^{1/2}$$

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72[I].—HERBERT E. SALZER & GENEVIEVE M. KIMBRO, *Tables for Bivariate Osculatory Interpolation Over a Cartesian Grid*, Convair-Astronautics, Convair Division of General Dynamics Corporation, San Diego, California, 1958, 40 p.

Formulas are developed for binary polynomials $P(x, y)$ which agree together with the partial derivatives $P_x(x, y)$ and $P_y(x, y)$, with $f \equiv f(x, y)$, $f_x \equiv f_x(x, y)$ and $f_y \equiv f_y(x, y)$ at n specified points. They have the advantage over ordinary bivariate interpolation of packing $3n$ conditions into n points. Unlike univariate polynomial osculatory interpolation which always possesses a solution for any irregular configuration of fixed points, a binary polynomial of prescribed form may not satisfy those $3n$ conditions for *any* choice of interpolation points, or may fail for just certain *special* configurations. Explicit formulas or methods are developed for the general 2- to 5-point cases. For interpolation over any square Cartesian grid of length h, for suitable 2- to 5-point configurations of (x_i, y_i), according to the formula

$$\text{(I)}\quad \begin{aligned} f(x, y) &\equiv f(x_0 + ph, y_0 + qh) \sim P(x_0 + ph, y_0 + qh) \\ &= \sum_{i=0}^{n-1} \{A_i^{(n)}(p, q)f_i + h[B_i^{(n)}(p, q)f_{x_i} + C_i^{(n)}(p, q)f_{y_i}]\}, \end{aligned}$$

tables of exact values of $A_i^{(n)}(p, q)$, $B_i^{(n)}(p, q)$ and $C_i^{(n)}(p, q)$ are given for p and q each ranging from 0 to 1 at intervals of 0.1. A closed expression for the remainder in (I) has not been found. In its place, formulas are derived for the leading terms in the bivariate Taylor expansions for the remainders. These formulas should cut down the number of needed strips in the numerical solution of Cauchy's problem for first order partial differential equations by the method of characteristic strips.

AUTHOR'S ABSTRACT

73[K].—D. E. BARTON & F. N. DAVID, "A test for birth order effect," *Ann. Human Genetics*, v. 22, 1958, p. 250–257.

In an ordered sequence of trials it is known that there were r_1 occurrences and r_2 non-occurrences of a particular event. The question at issue has to do with the randomness of the positions of the occurrences in the sequence vs. a tendency to appear either at the ends or in the middle of the sequence. A test criterion is obtained by dividing the sequence between the Rth and the $(R + 1)$st event, where $r_1 + r_2 = 2R$ or $2R + 1$, and then assigning ranks $1, 2, \cdots$ to the events by order of position beginning at the point of division and proceeding to the left and then again starting with 1 to the right. The sum of the ranks of occurrences as assigned is the test criterion S. For $r_1 + r_2 = 4(1)16$ with $r_1 \geqq r_2$, on the null hypothesis of random position of occurrences, the exact distribution of S is tabulated for each pair (r_1, r_2) for $r_2 \geqq 2$. For $r_1 + r_2 > 16$, it is stated that the distribution of S is sufficiently closely approximated by a normal distribution.

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74[K].—G. D. Berndt, "Power functions of the gamma distribution," *Ann. Math. Stat.*, v. 29, 1958, p. 302–306.

If x is a random variable from a gamma distribution with frequency function, $f_0 = f(x; \beta, \nu) = |\beta^\nu\Gamma(\nu)|^{-1}x^{\nu-1}\exp(-x/\beta)$; $\beta > 0$, $\nu > 0$ and $x \geqq 0$; the frequency function for δx with $\delta > 1$ is $f_1 = f(\delta x; \delta\beta, \nu)$. To test the null hypothesis on the mean, $H_0: \mu = \beta\nu$, against the alternate, $H_1: \mu = \delta\beta\nu$, $\delta > 1$, one may use the statistic $\alpha(x)$ with the critical region defined by $\alpha = \int_{\alpha(x)}^{\infty} f_0\, dx$. Then the power of this test is $\pi_\delta = \int_{\alpha(x)}^{\infty} f_1\, dx$. The mean of a random sample from a universe whose frequency law is $f_0 = f(x; \beta, \nu)$ obeys a gamma distribution with parameters β/n and $n\nu$. For $\alpha = .01, .05, .1$, charts are given for reading π_δ for $1 \leqq \delta \leqq 4$ and $\nu = \frac{1}{2}, 1(1)5, 7, 10(5)50$.

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75[K].—H. Chernoff & L. E. Moses, *Elementary Decision Theory*, John Wiley & Sons, Inc., New York, 1959, xv + 364 p., 24 cm. Price $7.50.

This book is an elementary approach in the theory of statistics through the theory of the strategy of games, and as such is a refreshing change from the usual run of elementary statistics textbooks. The authors state that only an understanding of high school (U.S.) mathematics is required, which is possibly optimistic. However, it is fair to say that the mathematical content of the book is not excessive, the exposition being mostly by example.

Chapter I gives the principles of decision and an introduction to minimax. Chapter II, entitled Data Processing, turns out to be our old friends graphical representation and means and standard deviations. No mention is made of grouping corrections. There are 38 pages on probability and random variables, both continuous and discrete, followed by a brisk treatment of utility and descriptive statistics. This chapter (IV) will be rather difficult for the beginner.

The authors have now reached a stage where they can, and do, begin to discuss strategies. Chapter V, "Uncertainty due to Ignorance of the State of Nature," gives simple Bayes strategies, minimax, and expected regret. (The reviewer liked the remark, "it is difficult to visualize four-dimensional space.") Further chapters cover further Bayes strategy and the application to problems which might arise in what is termed "Classical" statistics, in testing hypotheses, and in estimation. There is a series of appendices in which some of the statements in the main body of the text are proved.

This is an interesting book and may prove useful to those who see the interpretation of numerical data as just one more decision to take; it is greatly to be doubted whether it is of general utility. In one way, however, it is unique. Fisher and Neyman in their several ways might be said to have contributed to statistical decision theory, but are not deemed worthy of reference.

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76[K].—R. DOORNBOS & H. J. PRINS, "On slippage tests. I," *Indagationes Mathematicae*, v. 20, 1958, p. 38–46 (*Proc. Kon. Ned. Ak. van Wetensch.*, v. 61, Sec. A, 1958, p. 38–46); "On slippage tests. II," Ibid., p. 47–55; "On slippage tests. III," Ibid., p. 438–447.

The tables, which appear in part III, are related to two of the special cases included in this series of papers. In the first, from each of k Poisson distributions, with means μ_i, a random drawing Z_i is taken $(i = 1, 2, \cdots k)$. To test the null hypothesis that $u_i = u$, $i = 1, \cdots, k$, for which the table is prepared, against the alternate that one of the u_i's is greater than the others which have equal values, the authors propose the statistic, $\max Z_i$. For $k = 2(1)10$ and the sum of the k observations, $n \doteq 2(1)25$, values of $\max Z_i$ are given for which the significance levels are near 5% and 1%. In each case the actual significance levels are given to 3D.

In the second case, each of k objects is ranked by each of m observers. The null hypothesis under test is that each of the m rankings is independently and randomly chosen from the set of permutations of the integers $1, 2, \cdots, k$. As a test against the alternate that one of the objects has a higher probability of being ranked low while the others are ranked in random order, the proposed statistic is $\min S_i$ where S_i is the sum of ranks assigned the i-th object $(i = 1, 2, \cdots, k)$. Critical values S_α of $\min S_i$ for significance levels near $\alpha = .05, .025, .01$ are tabled for $m = 3(1)9$ and $k = 2(1)10$. Again in each case true significance levels are shown to 3D.

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77[K].—F. G. FOSTER, "Upper percentage points of the generalized beta distribution. III," *Biometrika*, v. 45, 1958, p. 492–503.

Let $\theta_{\max}$ denote the greatest root of $| \nu_2 B - (\nu_1 A + \nu_2 B)| = 0$ where A and B are independent estimates, based on ν_1 and ν_2 degrees of freedom, of a parent dispersion matrix of a four-dimensional multinormal distribution. Define

$$I_x(4; p, q) = \Pr(\theta_{\max} \leq k)$$

with $p = \frac{1}{2}(\nu_2 - 3)$, $q = \frac{1}{2}(\nu_1 - 3)$. Employing methods similar to those used in two preceding papers [1], [2] for the two and three-dimensional cases, the author tabulates 80%, 85%, 90%, 95%, and 99% points of $I_x(4; p, q)$ to 4D for $\nu_1 = 5(2)195$ and $\nu_2 = 4(1)11$.

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1. F. G. FOSTER & D. H. REES, "Upper percentage points of the generalized beta distribution. I," *Biometrika*, v. 44, 1957, p. 237–247. [*MTAC*, Rev. 165, v. 12, 1958, p. 302]
2. F. G. FOSTER, "Upper percentage points of the generalized beta distribution. II," *Biometrika*, v. 44, 1957, p. 441–453. [*MTAC*, Rev. 167, v. 12, 1958, p. 302.]

78[K].—W. HETZ & H. KLINGER, "Untersuchungen zur Frage der Verteilung von Objekten auf Plätze," *Metrika*, v. 1, 1958, p. 3–20.

For the classical distribution problem in which k indistinguishable objects are randomly distributed into n distinguishable cells (as in Maxwell-Boltzmann

statistics) the authors take the number, s, of occupied cells as a statistic to test the hypothesis of uniform probability over the cells. Let $P(s \mid n, k)$ be the probability density for s. The correspondence is noted between this distribution and the results of a series of n drawings from a discrete distribution in which the random variable assumes only the values $0, 1, 2, \cdots$, and in which the sample sum is k and the number of non-zero values is s. In developing a recursion formula for $P(s \mid n, k)$ it is shown that the uniform distribution over cells arises from the Poisson distribution, and the binomial and negative binomial distribution give particular non-uniformities. The function tabulated is $Z_{k;\alpha}$, which is defined under the hypothesis of uniformity by $\sum_{s=1}^{Z_{k;\alpha}} P(s \mid n, k) \leq \alpha$ and $\sum_{s=1}^{Z_{k;\alpha}+1} P(s \mid n, k) > \alpha$, for $\alpha = .05, .01, .001$; $n = 3(1)20$, and ranges of k varying from (3, 15) for $n = 3$ to (2, 100) for $n = 20$.

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79[K].—A. Huitson, "Further critical values for the sum of two variances," *Biometrika*, v. 45, 1958, p. 279–282.

Let s_i^2, $i = 1, 2$, be an estimate of the variance σ_i^2 with f_i degrees of freedom so that $f_i s_i^2/\sigma_i^2$ is distributed as χ^2 with f_i dif. To assign confidence limits to the form $\lambda_1\sigma_1^2 + \lambda_2\sigma_2^2$, where λ_1 and λ_2 are arbitrary positive constants, the author has previously [1] tabulated upper and lower 5% and 1% critical values of

$$(\lambda_1 s_1^2 + \lambda_2 s_2^2)/(\lambda_1\sigma_1^2 + \lambda_2\sigma_2^2).$$

The present tables are an extension, giving upper and lower $2\frac{1}{2}$% and $\frac{1}{2}$% critical values for the same function to 2D for $\lambda_1 s_1^2/(\lambda_1 s_1^2 + \lambda_2 s_2^2) = 0(.1)1$ and $f_1, f_2 = 16, 36, 144, \infty$.

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1. A. Huitson, "A method of assigning confidence limits to linear combinations of variances," *Biometrika*, v. 42, 1955, p. 471–479. [*MTAC*, Rev. 19, v. 12, 1958, p. 71.]

80[K].—Solomon Kullback, *Information Theory and Statistics*, John Wiley & Sons, New York, 1959, xvii + 395 p., 24 cm. Price $12.50.

This interesting book, which discusses logarithmic measures of information and their applications to the testing of statistical hypotheses, contains three extended tables in addition to a number of shorter or more specialized ones. Table I gives $\log_e n$ and $n \log_e n$ to 10D for $n = 1(1)1000$. Table II lists values of

$$p_1 \log_e \frac{p_1}{p_2} + (1 - p_1) \log_e \frac{1 - p_1}{1 - p_2} \quad \text{to 7D for} \quad p_1, p_2 = .01(.01).05(.05).95$$

(.01).99. Table III gives 5% points for noncentral χ^2 to 4D with $2n$ degrees of freedom for $n = 1(1)7$ and noncentrality parameter β^2 for $\beta = 0(.2)5$. As it is stated, this is taken directly from an equivalent table of R. A. Fisher [1].

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1. R. A. FISHER, "The general sampling distribution of the multiple correlation," *Proc. Roy. Soc., A.*, 1928, p. 654–673. See p. 665.

81[K].—G. J. LIEBERMAN, "Tables for one-sided statistical tolerance limits," *Industrial Quality Control*, v. 14, No. 10, 1958, p. 7–9.

Given a sample of n from $N(\mu, \sigma^2)$, it is desired to determine from the sample a quantity a (or b) such that with probability γ, the interval $(-\infty, a)$ (or the interval (b, ∞)) will include at least the fraction $1 - \alpha$ of the population. The tables give values of K to 3D for $n = 3(1)25(5)50$, $\gamma = .75, .9, .95, .99$, and $\alpha = .25, .1, .05, .01, .001$, such that $a = \bar{X} - Ks$ and $b = \bar{X} - Ks$, where X is the sample mean and S^2 is the usual unbiased estimate of σ^2. For more extensive tables and a more complete discussion see [1].

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1. D. B. OWEN, *Tables of Factors for One-sided Tolerance Limits for a Normal Distribution*, Office of Technical Services, Dept. of Commerce, Washington, D. C., 1958. [See RMT 82.]

82[K].—D. B. OWEN, *Tables of Factors for One-sided Tolerance Limits for a Normal Distribution*, Sandia Corporation, SCR-13, 1958, 131 p., 28 cm. Obtainable from the Office of Technical Services, Dept. of Commerce, Washington 25, D. C. Price $2.75.

Given a sample of n from $N(\mu, \sigma^2)$, with $\bar{x}$ the sample mean and S^2 the usual unbiased estimate of σ^2, these tables give values of k for which

$$\Pr[\Pr(x \leqq \bar{x} + ks) \geqq P] = \gamma.$$

As stated, Table I is a reproduction of one given by Johnson & Welch [1] in which values of k are given to 3D for $\gamma = .95$, $n = 5(1)10, 17, 37, 145, \infty$ and $P = 0.7(.05).85, .875, .9, .935, .95, .96, .975, .99, .995, .996, .9975, .999, .9995$. It is also explained that Table II was obtained from Resnikoff & Lieberman's table of percentage points of the noncentral t-distribution [2] appropriately modified to give k values to 3D for $n = 3(1)25(5)50, \infty$ and $P = .75, .85, .9, .935, .96, .975, .99, .996, .9975, .999$ for $\gamma = .75, .9, .95$. For $\gamma = .99, .995$, $n = 6(1)25(5)50, \infty$, while P has the same range as before. The more extensive Table III gives values to 5D obtained by an approximative method due to Wallis [3] for $n = 2(1)200(5)400(25)1000, \infty$, $P = .7, .8, .9, .95, .99, .999$, and $\gamma = .7, .8, .9, .95, .99, .999$. For small n and the larger values of P and γ, the approximation breaks down and the entry is left blank or given with a warning sign that comparison should be made with neighboring values. (However it looks to the reviewer as if this sign has been omitted from the entries for $n = 2$, $P = .99, .999$, and $\gamma = .999$.) Finally Table IV is obtained from Bowker's table of two-sided tolerance limits [3] by an approximate procedure suggested by McClung [4] to give conservative values of k for one-sided limits. Here values are given to 3D for $n = 2(1)102(2)180(5)300(10)400(25)750(50)1000, \infty$, $P = .875, .95, .975, .995, .9995$, and $\gamma = .75, .9, .99$.

In an appendix auxiliary tables compare values in the four tables for selected values of the four parameters. The maximum difference shown between Tables I and II is .01. It is concluded that values in Table III will probably be underesti-

mates for $\gamma \leqq .95$ and overestimates for $\gamma \geqq .99$, while in Table IV, k is probably underestimated for $P = .875$ and overestimated for the other P values. Differences shown between Table II and Table III values in a few cases exceed 20% of the presumably more accurate Table II values and differences shown between Table II and Table IV sometimes exceed 10% of the Table II values.

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1. N. L. JOHNSON & B. L. WELCH, "Applications of the non-central t-distribution," *Biometrika*, v. 31, 1939, p. 362–389.
2. G. J. RESNIKOFF & G. J. LIEBERMAN, *Tables of the Noncentral t-Distribution*, Stanford University Press, Stanford, Calif., 1957.
3. C. EISENHART, M. W. HASTAY & W. A. WALLIS, *Techniques of Statistical Analysis*, McGraw-Hill Book Co., New York, 1947.
4. R. M. MCCLUNG, "First aid for pet projects injured in the lab or on the range or what to do until the statistician comes," U. S. Naval Ordnance Test Station Technical Memorandum No. 1113, October 1955.

83[K].—K. V. RAMACHANDRAN, "On the Studentized smallest chi-square," Amer. Stat. Assn., *Jn.*, v. 53, 1958, p. 868–872.

Consider the F statistics, $\frac{S_i}{S} \cdot \frac{m}{t}$, $i = 1, 2, \cdots, k$, in which $S_1, S_2, \cdots, S_k$ and S are mutually independent, with each $S_{i/\sigma}2$ having a χ^2 distribution under the null hypothesis with t degrees of freedom and S/σ^2 a χ^2 distribution with m d.f. There are numerous applications of statistical methods, a few of which are discussed, in which one needs the value of V for which $\Pr \mid \frac{S_{\min}}{S} \frac{m}{t} \geqq V \mid = 1 - \alpha$. The author tabulates lower 5% points of $\frac{S_{\min}}{S} \cdot \frac{m}{t}$ for values of t, m and k as follows: For $t = 1, m \geqq 5, k = 1(1)8$ to 1S; for $t = 2, 5 < m < 10$ and $m \geqq 12$, $k = 1(1)8$ to 3D; for $t = 3, 4, 6$, $m = 5, 6(2)12, 20, 24, \infty$, $k = 1(1)8$ to 3D; for $t = 1(1)4(2)12, 16, 20$, $m = \infty$, $k = 1(1)8$ to 3D; for $t = 1(1)4(2)12, 16, 20$, $m = 5, 6(2)12, 20, 24, \infty$, $k = 1, 2, 3$ to 3D.

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84[K].—A. E. SARHAN & B. G. GREENBERG, "Estimation of location and scale parameters by order statistics from singly and doubly censored samples. Part II.," *Ann. Math. Stat.*, v. 29, 1958, p. 79–105.

This paper, a continuation of a previous one [1], is mainly devoted to an extension of tables given in the earlier paper to cover samples $11 \leqq n \leqq 15$ and to a discussion of efficiencies of the estimators used. Samples of n are from $N(\mu, \sigma^2)$; r_1 and r_2 observations are censored in the left and right tails respectively ($r_1 r_2 \geqq 0$); and $\bar{x}$ and σ are estimated by the most efficient linear forms in the ordered uncensored observations. Table I gives the coefficients for these best linear systematic statistics to 4D for all combinations of r_1, r_2 for $n = 11(1)15$. Table II gives variances and the covariance of these estimates to 4D for $n = 11(1)15$ and all pairs of r_1, r_2 values. In Table III efficiencies of the two estimates relative to that for uncensored samples are given to 4D for the same range of values of n and r_1, r_2. For

$n = 12$ and 15, variances and efficiencies relative to best linear systematic estimates are given for alternate estimates proposed by Gupta [2] for $n > 10$, and generalized in [1] to doubly censored samples, are given to 8D and 4D respectively for all r_1, r_2. The authors state that extensions of Tables I, II, III to 8D for $16 \leq n \leq 20$ are available upon application.

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1. A. E. SARHAN & B. G. GREENBERG, "Estimation of location and scale parameters by order statistics from singly and doubly censored samples. Part I. The normal distribution up to samples of size 10," *Ann. Math. Stat.*, v. 27, 1957, p. 427–451. [*MTAC*, Review 141, v. 12, 1958, p. 289.]
2. A. K. GUPTA, "Estimation of the mean and standard deviation of a normal population from a censored sample." *Biometrika*, v. 39, 1952, p. 88–95.

85[K].—J. M. SENGUPTA & NIKHILESH BHATTACHARYA, "Tables of random normal deviates," *Sankhya*, v. 20, 1958, p. 250–286.

As explained by the editor in a foreword, this is a reissue of an original table of random normal deviates which appeared in 1934 in *Sankhya* [1]. Since errors had been discovered in the earlier tables, the new set was reconstructed by conversion of Tippett's random numbers [2] to random normal deviates, as was the case before. After the present table was prepared, in 1952, as stated by the editor, it was learned that an identical table had been constructed in 1954 at the University of California. On comparison it was found that the two tables checked perfectly. As discussed in the text, rather extensive tests of the hypothesis that the entries were random drawings from $N(0, 1)$ were applied with satisfactory results. These tables contain 10,400 3D numbers.

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1. P. C. MAHARANOBIS, S. S. BOSE, P. R. ROY & S. K. BANNERJEE, "Tables of random samples from a normal population," *Sankhya*, v. 1, 1934, p. 289–328.
2. L. H. C. TIPPETT, *Random Sampling Numbers*, Tracts for Computers, No. XV, Cambridge University Press, London, 1927.

86[K].—MINORU SIOTANI, "Note on the utilization of the generalized Student ratio in analysis of variance or dispersion," *Ann. Inst. Stat. Math.*, v. 9, 1958, p. 157–171.

In samples from a p-dimensional normal universe an important statistic, applications of which are discussed in this paper, is $T_0^2 = m \operatorname{tr} L^{-1}V$ in which L and V are two independent unbiased estimates of the population variance matrix with n and m degrees of freedom respectively. Tables are given for the 5% and 1% points of the distribution of T_0^2 to 2D for $m = 1(1)10(2)20$ and

$$n = 10(2)30(5)50, 60, 80, 100.$$

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87[K].—Minoru Siotani & Masaru Ozāwa, "Tables for testing the homogeneity of k independent binomial experiments on a certain event based on the range," *Ann. Inst. Stat. Math.*, v. 10, 1958, p. 47–63.

Let k series of N trials each of a certain event be performed with the outcome of ν_i occurrences in the i-th series in which the fixed probability of occurrence was p_i, $i = 1, 2, \cdots, k$. To test the null hypothesis of homogeneity:

$$p_1 = p_2 = \cdots = p_k = p,$$

Siotani had previously proposed the statistic, $R_k(N, p)$, the range of the ν_i [1]. The tables in this paper give for $N = 10(1)20, 22, 25, 27, 30$; $k = 2(1)15$;

$$p = .1(.1).5;$$

$\alpha = .001, .005, .01(.01).06, .08, .1$, the greatest r_k for which

$$\Pr\{R_k(N, p) \geqq r_k\} < \alpha + .0005.$$

The cases in which for the r_k given, $\alpha < \Pr\{R_k(N, p) \geqq r_k\} < \alpha + .0005$ or

$$\alpha - .005 < \Pr\{R_k(N, p) \geqq r_k\} < \alpha$$

are indicated by attaching a + or a − respectively to the value of r_k.

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1. Minoru Siotani, "Order statistics for discrete case with a numerical application to the binomial distribution," *Ann. Inst. Stat. Math.*, v. 8, 1956, p. 95–104.

88[K].—P. N. Somerville, "Tables for obtaining non-parametric tolerance limits," *Ann. Math. Stat.*, v. 29, 1958, p. 599–601.

Let P be the fraction of a population having a continuous but unknown distribution function that lies between the r-th smallest and the s-th largest values in a random sample of n drawn from that population. Then for any $r, s \geqq 0$ such that $r + s = m$, Table I gives the largest value of m such that with confidence coefficient $\geqq \gamma$ we may assert that $100P\%$ of the population lies in the interval (r, s) for $\gamma = .5, .75, .9, .95, .99$ and $n = 50(5)100(10)150, 170, 200(100)1000$. Table II gives γ to 2D for the assertion that $100P\%$ of the population lies within the range, $(r, s = 1)$, in a sample of n for $P = .5, .75, .9, .95, .99$ and

$$n = 3(1)20, 25, 30(10)100.$$

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89[K].—G. P. Steck, "A table for computing trivariate normal probabilities," *Ann. Math. Stat.*, v. 29, 1958, p. 780–800.

Let X, Y, Z be standardized random variables obeying a trivariate normal dis-

tribution law. The author finds Pr $(X \leqq h, Y \leqq k, Z \leqq m)$ in terms of three functions:

$$G(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp\left(-\frac{x^2}{2}\right) dx, \qquad T(h, a)$$

$$= (2\pi)^{-1} \int_0^a [\exp\{-h^2(1 + x^2)/2\}](1 + x^2)^{-1}\, dx,$$

and

$$S(h, a, b) = \int_{-\infty}^{h} T(as, b) G'(s)\, ds.$$

The T-function has been tabulated by D. B. Owen [1, 2] and a table of $S(m, a, b)$ is given in the present paper to 7D for $a = 0(.1)2(.2)5(.5)8$, $b = .1(.1)1$ and a range of values of m decreasing from $0(.1)1.5$, ∞ for $a = 0(.1)1.2$ to $0(.1).3$, ∞ for $a = 6(.5)8$. The tabulated values are believed accurate to 0.6 in the seventh decimal place. There is considerable discussion of the main problem, of properties of and relations among the functions used, and a numerical example is worked out. The method of construction of the table is given and the efficacy of linear interpolation in it is discussed.

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1. D. B. OWEN, *The Bivariate Normal Probability Distribution*, Office of Technical Services, Department of Commerce, Washington, D. C., 1957, [*MTAC* Review 134, v. 12, 1958, p. 285–286.]

2. D. B. OWEN, "Tables for computing bivariate normal probabilities," *Ann. Math. Stat.*, v. 27, 1956, p. 1075–1090. [*MTAC* Review 135, v. 12, 1958, p. 286.]

90[K].—G. TAGUTI, "Tables of tolerance coefficients for normal populations," Union of Japanese Scientists and Engineers, *Reports of Statistical Application Research*, v. 5, 1958, p. 73–118.

The tolerance limits T_1, T_2 are to be determined so that with probability $1 - \alpha$ the interval (T_1, T_2) includes a given fraction, P, of the population. Following the method of Wald & Wolfowitz [1] for a sample from $N(\mu, \sigma^2)$, T_1 and T_2 are found by $T_1 = \hat{\mu} - k\sqrt{S_e/\nu}$ and $T_2 = \hat{\mu} + k\sqrt{S_e/\nu}$, in which $\hat{\mu}$ is an unbiased estimate of μ with variance σ^2/n and S_e is an independent error sum of squares with ν degrees of freedom. As illustrated by the author this permits useful applications in which n is not simply the sample size and $\nu = n - 1$ as is the case for the tables of Bowker [2]. The present tables give k to 3S for $P = .9, .95, .99$, $1 - \alpha = .9, .95, .99$, $n = .5(.5)2(1)10(2)20(5)30(10)60(20)100, 200, 500, 1000, \infty$ and $\nu = 1(1)20(2)30(5)100(100)1000, \infty$. The calculations were done with a slide rule and the author fears there may be errors up to one per cent. Some cursory comparisons with Bowker's tables for $\nu = n - 1$ showed frequent differences in the third significant figure.

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1. A. WALD & J. WOLFOWITZ, "Tolerance limits for a normal distribution," *Ann. Math. Stat.*, v. 17, 1946, p. 208–215.
2. CHURCHILL EISENHART, M. W. HASTAY & W. A. WALLIS, *Techniques of Statistical Analysis*, McGraw-Hill Book Co., New York. 1947. (See p. 102–107.)

91[K].—R. F. TATE & R. L. GOAN, "Minimum variance unbiased estimation for the truncated Poisson distribution," *Ann. Math. Stat.*, v. 29, 1958, p. 755–765.

For a sample of n from a population with the density function, $e^{-\lambda}\lambda^x/(1 - e^{-\lambda})$, $x = 1, 2, \cdots$, i.e., a Poisson distribution truncated on the left at $x = 1$, the authors derive the minimum variance unbiased estimation of

$$\lambda : \bar{\lambda}_0(t) = \frac{t}{n}\left(1 - \frac{\mathfrak{S}_{t-1}^{n-1}}{\mathfrak{S}_t^n}\right) = \frac{t}{n}\, C(n, t),$$

in which t is the sample sum and $\mathfrak{S}_t^n$ is a Stirling number of the second kind. Using an unpublished table of F. L. Micksa [1] of $\mathfrak{S}_t^n$ for $n = 1(1)t$, $t = 1(1)50$, this paper contains a table of $C(n, t)$ to 5D for $n = 2(1)t - 1$, $t = 3(1)50$.

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1. FRANCIS L. MIKSA, *Stirling numbers of the second kind*, RMT 85, *MTAC* v. 9, 1955, p. 198.

92[K, P].—P. A. P. MORAN, *The Theory of Storage*, John Wiley & Sons, Inc., New York, 1960, 111 p., 19 cm. Price $2.50.

This is a book about dams. Prof. Moran is at the Australian National University at Canberra, and I imagine that dams have great practical interest there. For many years he has been interested in estimating the probability that a dam will go dry or that it will overflow. He is also interested in how one finds a program of releasing water from a dam in such a way as to optimize the operations of a hydroelectric plant.

The first chapter contains some basic information about statistics and probability. To spare 14 pages for this from a total of a mere 96 shows how necessary Prof. Moran considered it to be.

The second chapter considers various general inventory and queueing problems analogous to dam problems.

In the third chapter the author plunges into his favorite topic, dams. First he considers discrete time—he looks at his water level only once a day. Under certain conditions distributions for the amount of water can be found, but two troublesome conditions occur which limit the regions of analyticity of the distributions. One is overflow. The other is running dry. If one ignores either or both of these, then he is dealing with an imaginary "infinite dam". Some queueing is analogous to an infinite dam, since there is no law limiting the lengths of queues.

Another chapter is devoted to dams which have as input a continuous flow, and from which the release is continuous.

In practice the inputs do not satisfy the assumption of independence, dry weeks tend to come in succession, so the results of the first four chapters are of limited applicability. Monte Carlo methods get estimates of the probabilities without

these restrictions, and in addition can be applied to configurations of dams completely beyond other methods of analysis. Of course Monte Carlo has disadvantages of its own. An example is given of a complex configuration for which probabilities were urgently wanted. A large retaining wall of earth was to be built. Overflow would ruin it, so a diversion tunnel was to be built large enough to insure against this contingency. During the building the tunnel is closed to permit pouring the concrete at its mouth. If water accumulates too high behind the wall there will be danger of overflow, ruining the wall. This can be prevented by opening the tunnel, ruining its outworks but preferable to damaging the main wall. The critical height changes each day as the wall is built up. What are the chances of this decision being forced?

The last few pages are devoted to ways of finding an optimum strategy for operating a hydroelectric system, or other program of releasing, replenishing, or otherwise tending the locks. The recommended solution is a method of successive approximations, which would probably be feasible only on a digital computer. The author suggests that a special analog device would be in order for the more complicated configurations.

The analogies between dams and queues or inventories are not pursued beyond the third chapter, in which it is merely mentioned. If these analogies are indeed valid they deserve more treatment. Without this treatment the title is misleading, for we find we are storing only water.

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93[M, X].—Jakob Horn & Hans Wittich, *Gewöhnliche Differentialgleichungen*, Walter de Gruyter & Co., Berlin, 1960, 275 p., 24 cm. Price DM 32.

This book is the sixth completely revised edition of Jakob Horn's *Gewöhnliche Differentialgleichungen*, which was published first in 1905. Like the previous editions, this book is intended for mathematicians, physicists, and engineers. In the selection of the material somewhat greater emphasis has been given to subjects that lend themselves to applications. Nevertheless, this book is primarily an introduction to the theory of ordinary differential equations. The text contains existence proofs and a comparatively detailed presentation of differential equations in the complex domain.

Considerable space is devoted to special functions which arise from differential equations. Numerical and graphical methods of solution are treated in a brief chapter. Besides a thorough knowledge of differential and integral calculus on the part of the reader, a familiarity with the basic concepts of the theory of functions of a complex variable is assumed.

No problem sections appear in the book, but numerous illustrative examples are provided.

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94 [Q, S].—Garrett Birkhoff and R. E. Langer, Editors, *Proceedings of Symposia in Applied Mathematics*, Vol. IX, "Orbit Theory," (Proceedings of the Ninth Symposium In Applied Mathematics of the American Mathematical Society, held at New York University April 4–6, 1957, cosponsored by The Office of Ordnance Research, Ordnance Corps, U. S. Army) American Mathematical Society, Providence, R. I., 1959, v + 195 p., 26 cm. Price $7.20.

The purpose of the book is, paraphrasing the words of the editors, to direct the attention of mathematicians to recent advances in celestial mechanics and, more importantly, to inform them of the problems that remain to be solved. Celestial mechanics owes its present form very largely to analysis, as it was developed in the eighteenth and nineteenth centuries. Whether modern mathematics can contribute anything important to the subject is a question that has hardly been explored, and it is high time that it should be.

Of the ten contributions by as many authors the first three deal with the motions of particles in magnetic fields, the remaining seven with motions of particles in gravitational fields. The magnetic fields considered are those in particle accelerators, in the galaxy, and about a laboratory model of the earth. The gravitational fields are principally those of the earth and of the solar system, although one paper deals generally with the field about any massive particle, and one with a general planetary system.

The various contributions are very uneven, ranging from rather trivial special applications of general formulae, through adaptations and modifications that are not trivial, to some important original contributions, both general and particular. Some authors describe what they have done themselves, some what others have done, and some what has not been done. Brouwer, Courant, and Olbert give special attention to unsolved problems; the references will be valuable to a mathematician not previously acquainted with their subjects. Herget and Eckert deal with practical computation.

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95[X].—Rudolph E. Langer, Editor, *Boundary Problems in Differential Equations*, Proceedings of a symposium conducted by the Mathematics Research Center, University of Wisconsin, Madison, Wisconsin, The University of Wisconsin Press, Madison, 1960, x + 324 p., 24 cm. Price $4.00.

This volume contains the nineteen papers presented at the symposium on 'Boundary Problems in Differential Equations" conducted by the Mathematics Research Center at Madison, Wisconsin during the period April 20–22, 1959. The papers are quite varied in nature and subject matter, as is clear from the table of contents given below:

Complete Systems of Solutions for a Class of Singular Elliptic Partial Differential Equations
Peter Henrici, University of California
Application of the Theory of Monotonic Operators to Boundary Value Problems
Lothar Collatz, University of Hamburg, Germany
Upper and Lower Bounds for Quadratic Integrals and, at a Point, for Solutions of Linear Boundary Value Problems
J. B. Diaz, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland
Error Estimates for Boundary Value Problems Using Fixed-Point Theorems
Johann Schroder, University of Hamburg, Germany
On a Unified Theory of Boundary Value Problems for Elliptic-Parabolic Equations of Second Order
Gaetano Fichera, The Mathematical Institute, University of Rome, Italy
Factorization and Normalized Iterative Methods
Richard S. Varga, Westinghouse Electric Corporation, Bettis Atomic Power Division, Pittsburgh
Some Numerical Studies of Iterative Methods for Solving Elliptic Difference Equations
David Young and Louis Ehrlich, The University of Texas.
Presented by David Young
Albedo Functions for Elliptic Equations
Garrett Birkhoff, Harvard University
A Numerical Method for Analytic Continuation
Jim Douglas, Jr., The Rice Institute, Texas
Stress Distribution in an Infinite Elastic Sheet with a Doubly-Periodic Set of Equal Holes
W. T. Koiter, Technical University, Delft, Holland
Some Stress Singularities and Their Computation by Means of Integral Equations
Hans F. Bueckner, Mathematics Research Center, U. S. Army
Boundary Value Problems in Thermoelasticity
Ian N. Sneddon, The Univerity, Glasgow, Scotland
Some Numerical Experiments with Eigenvalue Problems in Ordinary Differentia Equations
Leslie Fox, University Computing Laboratory, Oxford, England
Dynamic Programming, Invariant Imbedding, and Two-Point Boundary Value Problems
Richard Bellman, The Rand Corporation, California
Remarks about the Rayleigh-Ritz Method
Richard Courant, Institute of Mathematical Sciences, New York University
Free Oscillations of a Fluid in a Container
B. Andreas Troesch, Space Technology Laboratories, Inc., California
A Variational Method for Computing the Echo Area of a Lamina
Calvin H. Wilcox, Mathematics Research Center, U. S. Army, and California Institute of Technology

Workers in Numerical Analysis will be particularly interested in the papers

of Friedrichs, Garabedian, Collatz, Varga, Schroder, Young and Ehrlich, Douglas, and Fox. The first author has a very short paper in which he gives an interesting outline of a unified approach to the numerical treatment of linear partial differential equations irrespective of their type. The unified approach is said to also cover certain equations of mixed type. Unfortunately, the author did not have space to completely describe the conditions he must impose on the equations he treats.

Garabedian describes a method that has been used to calculate axially symmetric flows with free streamlines. In particular he discusses methods for calculating the contraction coefficient in the vena contracta. The method involves generalizing the differential equation governing the flow by introducing a parameter λ and studying the dependence of the solution as a function of λ.

Collatz's paper is an expository one in which he discusses various definitions of monotonic operators and applies such definitions to the determination of bounds on the solutions of various problems.

Schroder uses monotonic operators which satisfy a fixed point theorem to prove the existence of solutions to problems involving differential equations and boundary conditions. He also determines approximate solutions and error bounds by solving a so-called comparison problem.

Varga discusses a class of iterative methods for solving a system of linear equations which depend on the direct solution of matrix equations of matrices more general than tridiagonal matrices. He shows how such matrix equations can be directly and efficiently solved and, in addition, applies standard methods for accelerating convergence.

Young and Ehrlich report on numerical experiments which attempted to determine the extent to which theoretical results on the rate of convergence of the successive over-relaxation method for solving linear equations and for the Peaceman-Rachford method would apply for non-rectangular regions. The theoretical results are known for the latter method only in the rectangular case. In nearly every case it was found that the number of iterations using the Peaceman-Rachford method was less than was required using the successive over-relaxation method. However, approximately three times as much computer time is required for a double sweep of the former method as is required for a single step of the latter method.

Douglas discusses the determination of an approximation of an analytic function of a complex variable inside the disk $0 \leqq |z| \leqq 1$ when bounds on the function and its first two derivatives are known and when approximate values of the functions are known at p points equidistributed on the circle $|z| = 1$. An estimate of the error of the approximation is also obtained.

Fox discusses a method for the determination of approximate proper values and proper solutions to single or systems of ordinary differential equations for which a reasonable approximation is already known for the proper value. The method involves the introduction of parameters such as initial values of the solution at one of the boundary points and the determination of improved values for these parameters by the Newton process. The method described is not new, but the applications made by Fox to fairly difficult problems give an impressive demonstration of its power.

Space limitations prevent the reviewing of the remaining papers in this volume.

They are of high quality. The organizers of the conference are to be congratulated on the papers solicited. The University of Wisconsin Press has produced a handsome volume by a photographic process which makes a very readable page. The relatively low cost of the volume is especially noteworthy.

A. H. T.

96[X].—W. L. Wilson, Jr., "Operators for solution of discrete Dirichlet and Plateau problems over a regular triangular grid," May 1959, 29 cm., 191 p. Deposited in UMT File.

These tables list to 10D coefficients of a matrix operator for conversion of boundary values over an equilateral triangle to a discrete harmonic function over a regular triangular grid of 190 points in this triangle [1]. Sixty-three boundary values are involved, of which the three at the vertices do not influence the interior values of the function. The tables are useful in the approximate numerical solution of the Laplace equation over this triangular region.

Solutions for smaller triangles have been placed in the UMT File by the same author [2].

Also included are tables giving 10D coefficients of the analog of the Douglas functional over this same grid. Specifically, these are coefficients of a quadratic form (using scalar multiplication) of vector functions from the grid points of the bounding equilateral triangle to some euclidean space such that the value of the form is the Dirichlet integral

$$D = \frac{1}{2} \int (E + G)\, d\sigma$$

where E and G are coefficients of the first fundamental form of the surface got by linear interpolation of the discrete harmonic vectors resulting from application of the operator described above to the boundary values. This is a discrete analog of the functional used by J. Douglas [3] in his solution of the Problem of Plateau; it has application in the approximate numerical solution of that problem.

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1. L. V. Kantorovich & K. I. Krylov (translated by Curtis D. Benster) *Approximate Methods of Higher Analysis*, Nordhoff, Gronigen, Interscience, New York, 1958, p. 187–188.
2. W. L. Wilson, Jr., "Tables of inverses to Laplacian operators over triangular grids," UMT File, MTAC, No. 58, v. XI, 1957, p. 108.
3. J. Douglas, "Solution of the problem of Plateau," Amer. Math. Soc. *Trans.*, v. 33, 1931, p. 263–321.

97[Z].—Jack Bonnell Dennis, *Mathematical Programming and Electrical Networks*, John Wiley & Sons, Inc., New York, 1959, vi + 186 p., 24 cm. Price $4.50.

As the title indicates, the purpose of this little monograph is to explore the relationships of general programming problems and corresponding electrical networks, with a view towards gaining physical insight and developing computational algorithms. The contents of the book essentially comprise the author's doctoral

dissertation in the department of Electrical Engineering at M.I.T. The pages are offset reproductions of typescript. In a foreword by J. A. Stratton, it is stated that "there has long been a need for publication of research studies larger than a journal article but less ambitious than a finished book," and with this apology the present volume is put forth.

After an introductory Chapter 1, the general programming problem is presented in Chapter 2, along with discussions of convexity and concavity, the generalized method of Lagrangian multipliers due to Kuhn and Tucker, and duality. Chapter 3 consists of basic material on electrical networks containing resistors, diodes, ideal transformers, voltage sources, and current sources. The electrical network problem, which is a set of linear equations with side conditions in the form of inequalities due to the presence of diodes, is stated and shown to be equivalent to a quadratic programming problem (and its dual), viz., to find a feasible current distribution which minimizes the power absorbed by the voltage sources plus one-half the power absorbed by the resistors, with a corresponding statement for the dual. The concept of the two-terminal, or terminal-pair, network is introduced in this chapter, with a discussion of the set of solutions (ϵ, δ), where ϵ is the voltage between the terminals when current δ enters one terminal and leaves the other. The set of all (ϵ, δ) forms a "break-point curve" in the $\epsilon\delta$ plane, i.e., a non-decreasing polygonal graph.

Chapter 4 is devoted to the problem of flow in a network, which includes allocation, distribution, and assignment problems. It is shown that every flow problem can be realized by an electrical network containing only diodes, voltage sources, and current sources. Existence conditions from the theory of programming and non-redundancy assumptions are stated here in electrical network terms. Two algorithms are presented for the solution of diode-source networks, one corresponding to the primal and the other to the dual problem. They are similar to but more general than the procedure of Ford and Fulkerson for the transportation problem.

Chapters 5 and 6 treat the general linear and quadratic programming problems. A procedure is described for "tracing" the break-point curve corresponding to a pair of terminals, which is very similar to what goes on in the simplex method of Dantzig. This procedure is used with electrical models of the general quadratic (including linear) problems, and two types of algorithms for their solution are described. Chapter 7 contains a brief and incomplete discussion of the general programming problem. An algorithm is proposed which is based on the method of steepest descents. The main body of text is followed by eight appendices in which proofs are given of the main theorems of programming.

There are several typographical errors of a minor nature. The number of statements that are only partially true seems to be about par for a book on a mathematical subject written by an engineer.

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TABLE ERRATA

286.—Carl Burrau, *Tafeln der Funktionen Cosinus und Sinus*, etc., Reimer, Berlin, 1907.

This table is one of the relatively few that give e^x at .001 interval of x for values of x in the neighborhood of 9. (The range covered is 8.000 (.001) 9.809.) The values of e^x are not all in one place; some are on pages 8–10, and the remainder are on page 44.

A complete check of the 7-figure values of e^x for x from 8.000 to 9.210 revealed 65 last-figure unit errors.

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287.—F. E. Fowle, *Smithsonian Physical Tables*, First Reprint of Eighth Revised Edition, The Smithsonian Institution, Washington, D. C., 1934.

On p. 68, in Table 35, the following corrections should be made: $J_0(.62)$ should read .906184 instead of .905184; and $J_0(1.89)$ should read .287631 instead of .286631.

Charles R. Sexton

288.—J. W. Head & W. P. Wilson, *Laguerre Functions: Tables and Properties*, Monograph No. 183 R, The Inst. of Elec. Engineers, London, 1956. See R 47, *MTAC*, v. 12, 1958, p. 89.

A partial examination of Table 1, which tabulates 4D values of Laguerre functions $\lambda_n(x)$ for $x = 0(.1)1(.2)3(.5)6(1)10$ and $n = 0(1)10$, has disclosed 14 last-figure errors and two errors in sign in tabular entries corresponding to $n = 5$, $x = 1.4$ and $n = 6$, $x = 1.2$. These last should read $-.0088$ and .0086, respectively.

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289.—Her Majesty's Nautical Almanac Office, *Interpolation and Allied Tables*, Her Majesty's Stationery Office, London, 1956.

On page 66, in Gregory's interpolation formula, at the end, *for* $- \cdots$ *read* $+ \cdots$.

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290.—Her Majesty's Nautical Almanac Office, *Subtabulation, A Companion Booklet to Interpolation and Allied Tables*, Her Majesty's Stationery Office, London, 1958.

On page 31, the tabular entry corresponding to $a = 0$, $b = 175$, $r = 5$ should read 45 instead of 25.

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291.—Philip M. Morse, *Vibration and Sound*, Second Edition, McGraw-Hill, New York, 1948.

The following corrections should be made in Table V (p. 444): the terminal digits of the 4D values for $J_0(6.0)$, $N_0(6.4)$, and $N_1(6.6)$ should be decreased by a unit; the final digits of the values for $J_0(8.0)$, $N_0(3.0)$, $N_0(5.2)$, and $J_2(5.8)$ should be increased by a unit. For $J_2(7.4)$ the approximation should read -0.2490 instead of -0.2487.

All of these errors are present in the corresponding table appearing on p. 333 of the first edition (1936).

Charles R. Sexton

292.—Louis Robin, *Tables Numériques des Functions Associées de Legendre*, Editions de la Revue d'Optique, Paris, 1959.

This table contains a number of entries which have defects due to printing difficulties. The publisher has made available a listing containing these errors. Copies may be obtained by writing to the author.

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293.—I. M. Ryshik & I. S. Gradstein, *Summen-Produkt-und Integral-Tafeln: Tables of Series, Products, and Integrals*, Deutscher Verlag der Wissenschaften, Berlin, 1957.

An examination of these tables has revealed the following errors, some of which can be traced to sources cited in the authors' list of references.

On p. 416 the final decimal shown in the 16D approximation to Euler's constant should read 9 (when rounded) instead of 5.

Page	Formula	For	Read
2	0.126	$k = 1$	$k = 0$
	0.131	$A_4 = \frac{19}{80}$	$A_4 = \frac{19}{120}$
7	0.234 2	$k = 0$	$k = 1$
	0.234 3	$k = 1$ $(2k + 1)^2$	$k = 0$ $(2k + 1)^3$
	0.234 4	$(2k - 1)^2$	$(2k - 1)^3$
24	1.216 2	$+ \frac{7x^4}{4!} - \cdots$	$- \frac{7x^4}{4!} + \cdots$
34	1.393 1 1.393 2	$\frac{2k}{m}\pi$	$\frac{2k}{n}\pi$
44	1.621 4	$x - nx$	$x - n\pi$
149	3.235 1	1.171 953 619 4	1.171 953 619 3
186	3.622 7	−1.171 953 619 35	−1.171 953 619 34
	3.622 8	−0.157 660 149 15	−0.157 660 149 17
274	6.129 1	$\sqrt[4]{2}$	$\sqrt{2}$
	6.129 3	$\frac{\pi}{18}$	$\frac{\pi}{12}$
301	6.338 5	$\Pi_{k=1}^{\infty}$	$\Pi_{k=1}^{3}$
303	6.342 2	$1 - \zeta(2n + 1)$	$\zeta(2n + 1)$
330	6.514 8 6.514 9	$Z_{p+\frac{1}{2}}ci(z)$	$Z_{p+\frac{1}{2}}(ciz)$

Page	Section	For	Read
413	8.21	$B_{22} = \frac{854\ 513}{123}$	$B_{22} = \frac{854\ 513}{138}$
		$B_{34} = \frac{2\ 577\ 867\ 858\ 367}{6}$	$B_{34} = \frac{2\ 577\ 687\ 858\ 367}{6}$
	8.3	$\zeta(11) = 1.000\ 494\ 183\ 6$	$\zeta(11) = 1.000\ 494\ 188\ 6$
414	8.41	0.154 398 101 8	0.154 981 017 1
		0.149 446 010 5	0.149 445 980 8
		0.144 479 346 3	0.144 464 448 1
		67 108 684	67 108 864
	8.42	0.007 312 527 5	0.007 312 525 9
		0.006 447 210 5	0.006 447 210 3
		0.005 740 026 5	0.005 740 037 7
		0.004 660 148 3	0.004 660 143 5
415	8.44	1125	1152
		0.000 229 601 1	0.000 229 601 5
		0.000 160 694 8	0.000 160 694 6

With the exception of the errors found on pages 7 and 44 and the error of transposition of digits detected on page 415, all the errors noted above appear also in the Russian third edition.

The errors noted in formulas 0.131 and 1.216 2 appear in both Adams [1] and Jolley [2], whereas the error noted in formulas 1.393 1 and 1.393 2 occur only in

the latter. The errors in formulas 6.129 1 and 6.129 3 apparently were reproduced from Magnus and Oberhettinger [3].

The terminal-digit errors occurring in definite integrals 3.235 1, 3.622 7, and 3.622 8 are due to unjustified retention of guard figures from data in Lindman [4], Table 113 (p. 61).

The tabular errata noted on p. 414–415 appear in all the previous editions of this book. The exact values of $(2n - 1)!!/(2n)!!$ and of $(2n - 1)!!/(2n + 2)!!$ for $n = 1(1)15$ and $n = 1(1)14$ were given by Lambert [5]. I have verified by independent calculation that all of Lambert's values are free from error.

Confused notation renders formula 6.362 (p. 307) incorrect. It should read

$$C = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6}$$

$$- \frac{1}{240n^8} + \cdots + \frac{B_{2r}}{2r}\frac{1}{n^{2r}} + \frac{B_{2r+2}}{2(r+1)}\frac{\theta}{n^{2r+2}} \qquad [0 < \theta < 1].$$

J.W.W.

1. E. P. Adams, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, The Smithsonian Institution, Washington, 1947.
2. L. B. W. Jolley, *Summation of Series*, Chapman & Hall, Ltd., London, 1925.
3. W. Magnus & F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics*, Chelsea Publishing Company, New York, 1954.
4. C. F. Lindman, *Examen des Nouvelles Tables d'Intégrales Définies de M. Bierens de Haan*, Stockholm, 1891. Reprinted by G. E. Stechert & Co., New York, 1944.
5. J. H. Lambert, *Zusätze zu den Logarithmischen und Trigonometrischen Tabellen zur Erleichterung und Abkürzung der bei Anwendung der Mathematik vorfallenden Berechnungen*, Berlin, 1770.

294.—L. Silberstein, *Synopsis of Applicable Mathematics with Tables*, Bell, London, also Van Nostrand, New York, 1923.

The following corrections should be made in the tables of Bessel functions, p. 143.

x	$J_0(x)$ For	$J_0(x)$ Read
.62	.90518	.90618
1.89	.28663	.28763
2.70	−.11424	−.14245
5.90	.11203	.12203
6.92	.29873	.29874
	$J_1(x)$	
5.87	−.30109	−.30019
6.32	−.20291	−.20292
7.87	.21401	.21407

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295.—G. W. Spenceley & R. M. Spenceley, ***Smithsonian Elliptic Functions Tables,*** **The Smithsonian Institution, Washington, D. C., 1947.**

The following two corrections should be made in the tabular values of ϕ appearing on page 187:

θ	r	For	Read
47°	36	0.72012 80371 23	0.72012 80370 23
47°	44	0.82824 23463 76	0.86314 89314 15

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